Properties of IRR Equation with Regard to Ambiguity of Calculating of Rate of Return and a Maximum Number of Solutions

IRR equation is widely used in financial mathematics for different purposes, such as computing rate of return on investment, calculation of implied volatility, yield to maturity, calculation of interest rates for mortgages and annuities, etc. However, in general, IRR equation may have several solutions, which restricts its application. Thus, the knowledge of how to find these solutions and choose the right one is important. Our understanding of properties of IRR equation is not complete. This article studies when solutions of IRR equation exist and what factors define the number of solutions. It also explores other properties of IRR equation with regard to the problem of calculating rate of return in general, and interest rate for mortgages and annuities.

Yuri Shestopaloff, Ph.D.
is a Director of Technology, Research & Development at SegmentSoft Inc. He is a Doctor of Sciences and has Ph.D. in mathematical methods for data processing and interpretation. He develops mathematical algorithms for the financial industry. He does mathematical modeling in biology, remote sensing, wave propagation theory. Yuri also designs middle and large-scale software applications and systems for financial and high tech companies, including performance measurement and algorithmic trading systems. He has published over eighty professional articles and ten books.

Wolfgang Marty, Ph.D.
is Vice President at Credit Suisse in Zurich. He specializes in performance attribution, fixed income in general and Modern Portfolio Theory. Previously, he was a director of Economic Research at UBS Warburg Dillon Read in London. He teaches courses in Quantitative Finance at various Swiss Universities, the CS Business School and the International Faculty of Finance. He publishes regularly in several scientific journals. He also chairs the Swiss Bond Commission and is a member of the Executive Committee of European Bond Commission. He holds a degree in mathematics from the Swiss Federal Institute of Technology and a Ph.D. from the University of Zurich.

GENERAL PROPERTIES OF IRR EQUATION

The problem of how many solutions the IRR equation has was studied based on different approaches (Eschenbach, 2007, Shestopaloff, 2009, 2010). In Shestopaloff, 2010, 2011, the author proved several theorems that can find the maximum number of possible solutions for the power equation (which is also called generalized polynomial equation), and for equations composed of sums of exponential functions. Through the series of transformations, any form of IRR equation can be rewritten as a power (generalized polynomial) equation, whose maximum possible number of solutions is equal to the number of sign changes of consecutive coefficients, similar to the well known Descartes Rule of Signs for the polynomial equations with integer powers. However, unlike the case of Descartes Rule of Signs, in the generalized polynomial equation, the powers can be any real number.

Below, we briefly analyze the properties of IRR equation that are related to its practical application for investment performance measurement. The properties of IRR equation were comprehensively studied in Shestopaloff, 2009. The IRR equation has different equivalent forms. All of them are derived from scratch in the aforementioned work. Let us consider the following traditional form.

$$E = B(1+R)^{T_0} + \sum_{j=1}^{N} C_j (1+R)^{T_j} \qquad (1)$$

Here, B is the beginning market value of a portfolio, which is positive by definition; E is the ending market value, which is non-negative; C_j is the cash flow that can be positive (if it is added to the portfolio), or negative (if the cash has been withdrawn from the portfolio); T_j is the time period from the origination of the cash flow until the end of the investment period; T_0 is the length of the overall period; j=1,…N; R is the rate of return that we want to find. Note that this is the rate of return that corresponds to *one unit* of time, which we use for measuring T_j.

IRR equation allows computing rate of return at any arbitrary moment of time, which does not necessarily coincide with cash transactions. Suppose, we want to know the rate of return for such an arbitrary period T_0. Cash transactions were made at the moments $t_1,...t_j,...,t_N$, where time is calculated from the beginning of the investment period. Then, the values T_j in (1) are calculated as follows: $T_j = T_0 - t_j$. The solution of (1) will be the rate of return for the total period T_0. The beginning of the investment period can be chosen at any arbitrary moment of time as well. Then, only cash transactions within this period have to be taken into account in (1).

The ending market value cannot be negative. This restriction implies a simple fact that the investor cannot lose more than he invested. For the same reason, the rate of return $R \geq -1$. We can add the first term in (1) to the sum represented by the second term. The form (1) pays tribute to the industry's convention, which traditionally distinguishes the beginning market value and cash flows. In fact, the beginning market value can also be considered as a cash flow that occurs at the beginning of the investment period. Thus, in the general case, equation (1) can be rewritten in a more compact and convenient form as follows.

$$E(R) = \sum_{j=0}^{N} C_j (1+R)^{T_j} \qquad (2)$$

Here, $C_0 = B$; the rest of values C_j are cash flows, the same as in (1). It is important to note that the terms in (2) have to be written in the descending order of powers, that is

$$T_0 > T_1 > ... > T_{j-1} > T_j > ...T_N \qquad (3)$$

The specifics of adding and withdrawing cash flows imply some mathematical restrictions compared to the general case of power equation. Namely, the withdrawal amount cannot be greater than the current market value of the portfolio. Also, the beginning market value C_0 has to be positive, and the ending market value E cannot be negative. Based on the results of the study in Shestopaloff, 2010, 2011, we can say that for the generalized polynomial equation (2) the maximum number of positive solutions is equal to the number of consecutive sign changes of the coefficients, or less than this number by a multiple of two. The number of negative roots is found in the same way by considering a similar equation obtained from (2) by changing the sign of variable $(1+R)$ (this procedure affects the odd powers only).

As an example, let us consider the following IRR equation with irrational powers.

$$(1+R)^{\sqrt{7}} - 2.5(1+R)^{\sqrt{5}} + 1.01(1+R)^{\sqrt{3}} + 0.99(1+R)^{\sqrt[3]{2}} = 0 \qquad (4)$$

In this equation, the terms are arranged in descending order of powers, $\sqrt{7}, \sqrt{5}, \sqrt{3}, \sqrt[3]{2}$. The equation's coefficients $1, -2.5, 1.01, 0.99$ change their algebraic sign two times: from 1 to -2.5, and from -2.5 to 1.01. So, (4) has a maximum of two positive solutions or none.

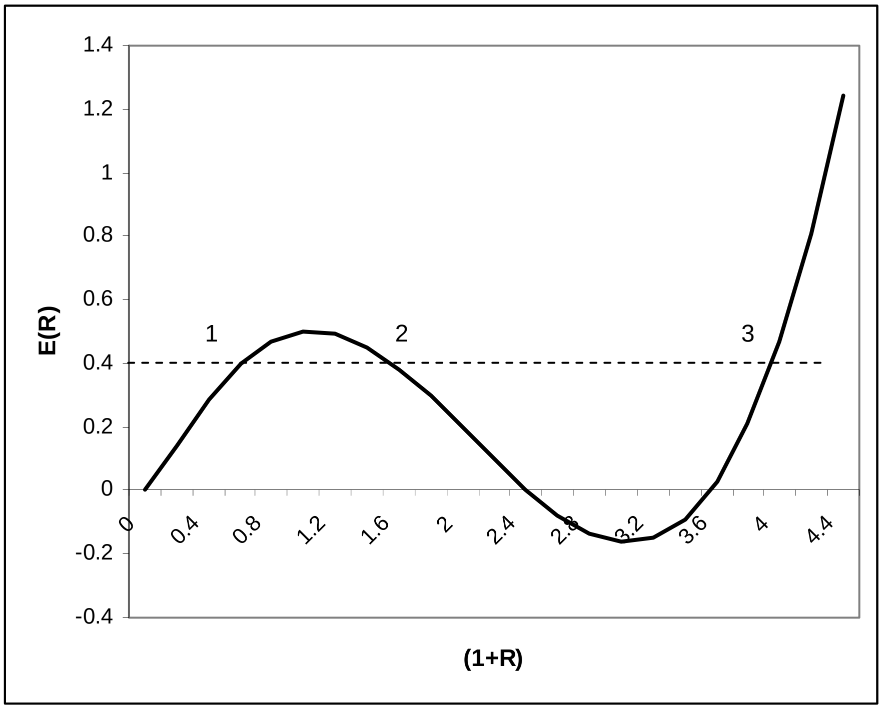


Figure 1: The Graph of Function *E(R)*

When the free term is zero, the appropriate equation has two solutions. When we add the free term of (-0.4), then the number of positive solutions increases to three because of the additional sign change between coefficients.

Figure 1 shows the graph of function $E(R)$ that represents the left side of (4).

$$E(R) = (1+R)^{\sqrt{7}} - 2.5(1+R)^{\sqrt{5}} + 1.01(1+R)^{\sqrt{3}} + 0.99(1+R)^{\sqrt[3]{2}} \qquad (5)$$

We can see that equation (4) has two positive solutions and also the zero solution (when $1+R=0$) because the graph of function in this case intersects the abscissa axis two times for positive values of $1+R>0$. If we add the negative free term of (-0.4), then (4) transforms to

$$(1+R)^{\sqrt{7}} - 2.5(1+R)^{\sqrt{5}} + 1.01(1+R)^{\sqrt{3}} + 0.99(1+R)^{\sqrt[3]{2}} - 0.4 = 0 \qquad (6)$$

We have an additional change of the algebraic sign between the equation's coefficients, is from 0.99 to (-0.4). So, this equation can have a maximum of three positive solutions, or one. In this case, we have three solutions that are shown in Figure 1. The zero solution disappeared.

This example shows that, without taking into account specifics of IRR equation, the number of possible solutions can be as many as the number of terms (if we include the zero solution), or, it can be equal to the total number of cash deposits and cash withdrawals, including the beginning market value.

Note that the powers in this perfectly legal IRR equation are irrational numbers. It cannot be converted to a polynomial equation. Irrationality of powers in IRR equation is possible because the time interval between the moment when the cash flow has been made and the end of the considered investment period can be any real number.

SPECIFIC FEATURES OF IRR EQUATION INFLUENCING THE NUMBER OF SOLUTIONS

We will now consider specific features of IRR equation. First of all, because of the non-negative value of $(1+R)$ in (2), we should consider only values of $R \geq -1$. It means that we should count only half of all possible solutions plus zero solution, when (2) does not a have a free term.

Another restriction is that $C_0 > 0$, which means, in IRR terms, the positive beginning market value. Also, from a practical perspective, the ending market value cannot be negative. Intermediate withdrawals also cannot make the market value of the portfolio negative. In fact, the market value of an investment portfolio cannot be negative at any moment. However, if we assume that at any given cash transaction the market value remains positive, then it cannot become negative in between cash transactions when the rate of return $R > -1$. Let us call such an investment portfolio an "always positive" portfolio.

For the "always positive" portfolios, besides the restriction that the beginning and the ending market values should satisfy accordingly, the conditions $C_0 > 0$ and $E \geq 0$, we may impose an even stronger and more general form of such a restriction that implies a non-negative market value of the portfolio at the moment of each cash transaction. However, the question remains, what value of rate of return should we use? One answer is that we assume that the rate of return is the same between cash transactions as it is for the whole investment period. In this case, the condition of non-negative value of the investment portfolio at the moment of cash transaction can be formulated as follows.

$$E_J(R_k) = \sum_{j=0}^{J} C_j (1+R_k)^{T_j - T_J} \geq 0 \qquad (7)$$

for all $J = 0,1,...,N$ and all real solutions of (2) $R_k > -1$, k=0,1,…,*K*.

In fact, (7) computes the market value of the portfolio at the moment of each cash transaction and checks if the market value is non-negative. The rate of return in between cash transactions is assumed the same. Note that the condition $E \geq 0$ is included in (7), and we replace the condition that the beginning market value of portfolio is positive by the condition that it is non-negative. When we have intermediate cash transactions, then a portfolio with a zero beginning market value makes sense. (Some financial instruments, such as annuities, may have a zero beginning market value.)

In the next scenario, the rate of return varies during the investment period and we know its values between cash transactions. In this case, the condition of non-negative market value of portfolio is defined by the following formula.

$$E_J = \sum_{j=0}^{J} C_j \prod_{m=j+1}^{J} (1+R_m)^{T_m - T_{m-1}} \geq 0, \quad \text{when } J \geq 1$$

$$E_J = C_0, \quad \text{when } J = 0 \tag{8}$$

Here, $J = 0,1,...,N$; R_m is the rate of return for one period between the $(m-1)$ -th and *m*-th cash transactions. This value of rate of return is for one unit of time, so that we have to use compounding in order to calculate the rate of return for the *m*-th period. The compounding operation is represented by the power $(T_m - T_{m-1})$.

For instance, (8) implies that we cannot make a withdrawal greater than the current market value of the portfolio. Although we consider (8) for discrete values of time, it certainly can be used for continuous time. In this case, the values of T_m have to be substituted by the value of time between the chosen cash transactions. This time has to be calculated from the last previous transaction. Certainly, indexes in (8) should be adjusted to the last considered period. The same is true for equation (7) which can also accommodate continuous time. Below, for convenience, we consider discrete forms of (7) and (8). However, we should keep in mind that the obtained results are valid for the continuous time as well. In this regard, conditions (7) and (8) are valid for an arbitrary period of time in the same way as the IRR equation is, as we discussed earlier.

Note that restriction (7) was obtained with the assumption that the rate of return is the same throughout the whole period. We will see later how the conditions (7) and (8) can be relaxed with regard to calculation of rate of return.

What is the practical value of “always positive” portfolios and restrictions (7) and (8)? They can be used in analytical studies when financial analysts do research of investment strategies and need to simulate different investment scenarios to see their timely progress. In such cases, the portfolio’s market value cannot be negative at any moment; otherwise the simulation will be invalid.

EXAMPLES

Let us consider a portfolio that has one cash flow C_1, in the middle of the investment period. In this case, we can rewrite (2) as a quadratic equation as follows.

$$E = C_0(1+R)^2 + C_1(1+R) \tag{9}$$

Let us substitute $x = 1 + R$. This implies that the domain $x > 0$ because $R \geq -1$. Then, we have

$$C_0 x^2 + C_1 x - E = 0 \tag{10}$$

The solution of (10) is

$$x_{1,2} = \frac{-C_1 \pm \sqrt{C_1^2 + 4C_0 E}}{2C_0} \qquad (11)$$

So, in this simple case, given the aforementioned restrictions applied to algebraic signs of the beginning and ending market values, the solution always exists because the discriminant $C_1^2 + 4C_0 E \geq 0$. The sign of cash flow does not matter. We can also see that one solution will always be negative. This solution can produce the rate of return less than minus one. For instance, if $C_0 = 3$, $C_1 = 4$, $E = 6$, then $x_1 \approx 0.897$, $x_2 \approx -2.23$, and accordingly $R_1 \approx -0.103$, $R_2 \approx -3.23$. Certainly, the second solution does not have business sense.

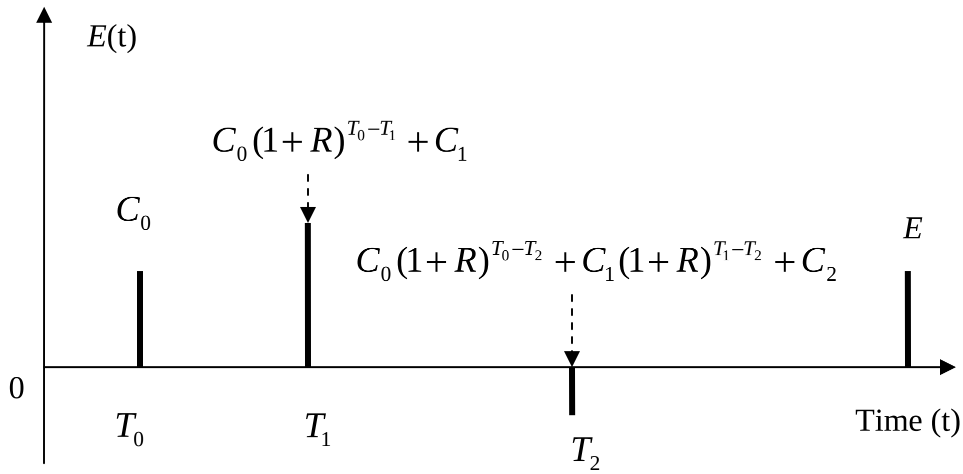
THE PROPERTY OF INTERMEDIATE MARKET VALUES AND THEIR RELATION TO VALUES OF CASH FLOWS

One of the interesting restrictions with regard to IRR equation that is defined by the specifics of the investment portfolio is that when we withdraw cash from the portfolio, the current market value has to be greater than the value of withdrawal at the moment of withdrawal. However, the investment process is not uniform. It may happen that during a short period of time the investment grows tenfold during several days while the total investment period is several years. Such a situation allows the investor to withdraw a substantial amount of money while leaving some amount in the portfolio. However, the investment may actually decrease, so that the total rate of return will be relatively low compared to the quick original growth at the beginning of investment period. In this case, when we apply the average rate of return calculated for the whole period, the intermediate market value of portfolio can be negative at the moment when the first large withdrawal has been made. For instance, let us consider the following numerical example when the appropriate IRR equation has three solutions. The parameters are as follows: $C_j = \{1.0, -3.0, -4.0, -3.0, 11.0\}$, $T_j = \{0.9, 0.8, 0.6, 0.5, 0.001\}$. The intermediate market values of the portfolio for the second solution at the moment of cash transaction are as follows: $E_j = \{1.0, -1.966, -6.102, -9.31, 0.0\}$. For the first and smallest solution, three intermediate market values are also negative, while all intermediate market values are positive for the third solution, although the value of this rate of return is about 270,000, which is not a practical one given the presented values of cash transactions. This computational aspect has been previously studied in Chestopalov, 2008 with regard to conceptual context of rate of return. This example, once again, confirms the importance of choosing the right investment *context*.

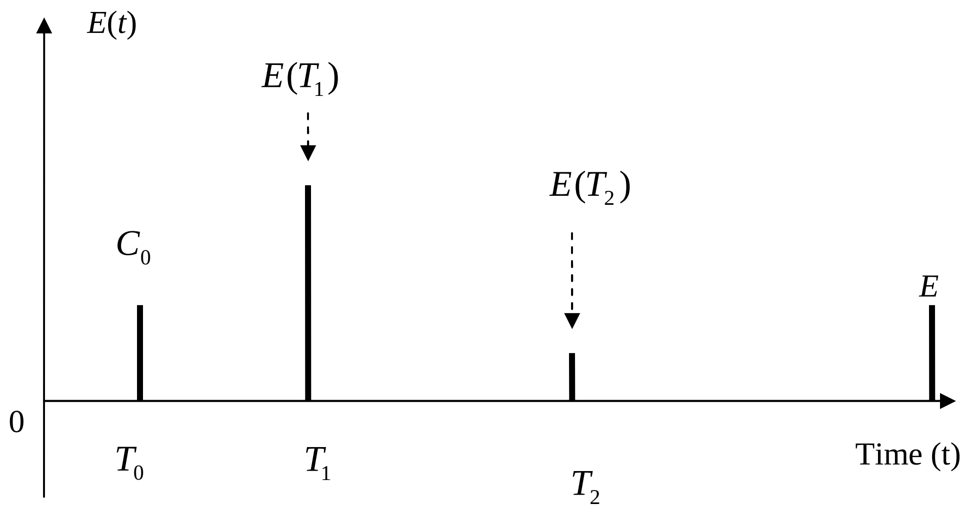
So, we have found an interesting property of IRR equation, which is the possible negative intermediate value of the current market value of portfolio when we use the average rate of return. This property is the result of the non-uniform growth of investment. If we assume that the investment grows at the same rate of return, then we should exclude all solutions which produce intermediate negative market values of portfolio.

EXAMPLE OF PORTFOLIO WITH NEGATIVE CASH FLOWS

Let us consider a hypothetical portfolio with two cash flows. Figure 2 illustrates this graphically. If we apply the rate of return for the whole period, then the current market value of the portfolio at the moment of the second cash transaction is negative (Figure 2a), while in reality it cannot be negative because the cash withdrawal cannot be made. In fact, during the period between T_0 and T_2, the actual rate of return was higher than the average rate of return for the whole period. The actual growth was higher than average, and the portfolio at the moment T_2 actually had a positive market value (Figure 2b). However, after T_2 and until the end of the investment period, the rate of return was lower than average for the whole period, which decreased the overall rate of return.



a



b

Figure 2: Change of an Intermediate Market Value of a Portfolio with Two Cash Flows Computed Using (7).

a) - intermediate market value is negative when the second cash flow C_2 occurs. b) – the actual intermediate market value of the portfolio is higher because of the higher than average rate of return at the beginning of the investment period between T_0 and T_1.

So, the condition (7) is not fulfilled, although the value of the average rate of return has a practical meaning. At the same time, the condition (8) remains true. So, depending on the business scenario, we can use both conditions. For instance, if we are going to use the average rate of return through the whole investment period, then (7) has to be fulfilled. Otherwise, there's a risk that the market value of portfolio will become negative. If we are only interested in the value of the rate of return for the whole period, then we can ignore this condition. If we know the rates of return between cash transactions, then (8) should be fulfilled. It can serve as a verification procedure to evaluate the validity of rates of return and exclude invalid solutions. The ambiguity and diversity of scenarios is not a convenient thing, but this is what we have to understand in order to use the IRR equation in practical business applications.

SPECIFIC PROPERTIES OF THE MORTGAGE IRR FUNCTION

In Shestopaloff, 2010 the mortgage IRR function was studied (which is the IRR function $E(R)=\sum_{j=0}^{N} C_j (1+R)^{T_j}$ rewritten for mortgages with equal payments). It was found that it always has one minimum. This allows a substantial reduction to the ambiguity of the IRR equation's solution. The graph of this function is presented in Figure3.

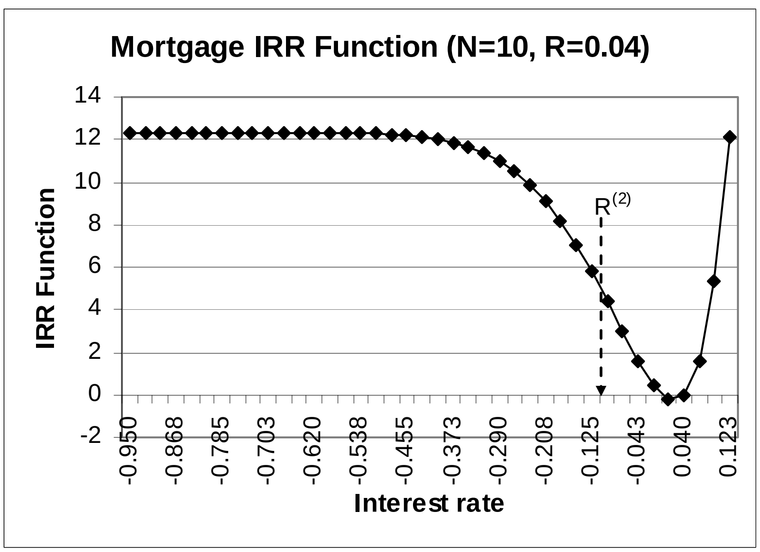


Figure 3: Graph of the Mortgage IRR Function
Number of periods is 10; rate of return per period is 4 %.

The mortgage IRR function $F(R)$ can be presented as follows Shestopaloff, 2010.

$$F(R) = P(1+R)^{n+1} - (P+C)(1+R)^n + C \tag{12}$$

where P is the principal mortgage amount; C is the regular payment; R is the interest rate for one period; n is the total number of periods.

The practical solution will always be located to the right of the minimum of IRR function. So, if we find the value of the interest rate corresponding to this minimum, then the domain to the right of this value of the interest rate will always have a unique solution. This interest rate can be found by equating the first derivative of the mortgage IRR function to zero and solving the obtained equation with regard to the interest rate. The first derivative can be found as follows.

$$F^{(1)}(R) = P(n+1)(1+R)^{n} - n(P+C)(1+R)^{n-1} \qquad (13)$$

Equating the first derivative to zero, we will find the following value.

$$R_0^{(1)} = \frac{nC - P}{(n+1)P} \qquad (14)$$

To the right of this point, the solution of mortgage equation (12) will always be unique. This is an important consideration because most often the IRR equation is solved by iterative algorithms. So, securing the value of the first iteration greater than the interest rate defined by (14) is important in order to obtain a correct value of interest rate.

FINDING AND CHOOSING THE RIGHT FROM THE BUSINESS PERSPECTIVE SOLUTION OF IRR EQUATION

Multiplicity of IRR equation solutions requires some criteria for the selection of the right solution that makes business sense. Figure 4 shows two typical graphs of IRR function. Based on the results of work Shestopaloff, 2009 and our modeling scenarios, the following approaches can be considered by financial analysts and system developers.

First of all, there are approximations of the IRR equation that have less or do not have solution ambiguity. For instance, the Linear approximation of IRR equation presented in Shestopaloff, 2009 and the Modified Dietz equation always have one solution. So in the situation where solutions obtained by the Linear approximation or Modified Dietz formula can be considered reliable, the closest solution of IRR equation should be considered as the right one. This is the case when cash transactions are relatively small compared to the market value of a portfolio, and the rates of return are either positive or slightly negative. For the negative rates of return that are close to minus one, the Modified Dietz equation produces large errors (Shestopaloff, 2009).

The second approach would be to use the solution of a quadratic approximation of IRR equation (Shestopaloff, 2009), which produces substantially more reliable results in the wider range of rates of return, including solutions close to minus one. Quadratic approximation can also be used as the first iteration in the iterative procedure for solving IRR equation, so that this way we avoid additional computational overhead related to finding the quadratic approximation. Overall, this is an efficient approach, that is finding an approximate rate of return by less ambiguous procedure first, and then choosing the closest solution of IRR equation if it has multiple solutions.

Another less accurate but simpler approach is this. If the total of deposits is D (which includes the beginning market value) is greater than the total of all withdrawals W plus the ending market value E, then the investment has most likely a negative rate of return. The estimation can be found as

$$R = (E - W - D) / D \tag{15}$$

Here, deposits are positive values, withdrawals are negative.

What we found in modeling and real scenarios is that the Linear and Modified Dietz approximations were unambiguous indicators of the right, from the business perspective, solution. When the IRR function “sinks” below the abscissa, then the first solution (point 3 in Fig. 4) was the right one, and the Linear approximation and Modified Dietz solutions were much closer to this value than to the others. The quadratic approximation, in all of its forms, produces unambiguous and reliable results, which are substantially closer to the solution of IRR equation than in case of Linear approximation or Modified Dietz formula. We should mention that sometimes the first solution of the quadratic approximation can be used, but this is the case of almost complete loss of investment, which can be easily identified.

So, if we use the above safeguards, that is the quadratic approximation, Linear approximation and Modified Dietz methods in order to find the initial approximate value of rate of return, then the exact solution can be found unambiguously, with a very rare exception when the solution is close to the value of rate of return $R = -1$, or solutions of IRR equation are extremely close, which is also a very rare situation that occurs when the *large* (compared to the beginning market value of portfolio) *and frequent* cash withdrawals *and* deposits are made I sequence and the rate of return is large, of the order of several hundred percent.

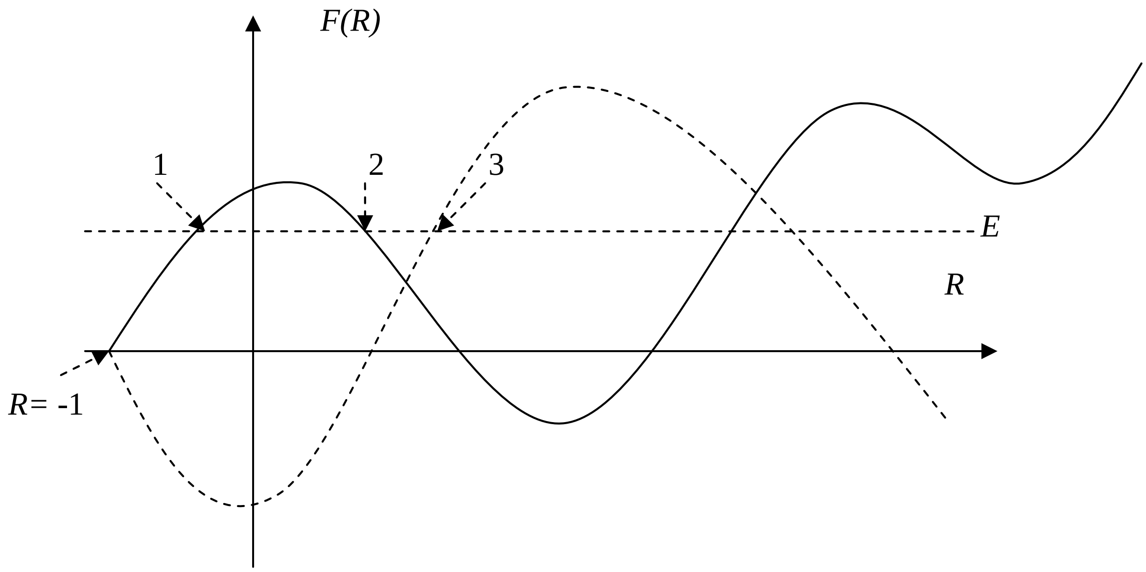


Figure 4: Graphs of IRR Function and Solutions of IRR Equation.

The following “rules of thumb” on how to choose the right solution, are listed in order of priority. In theory, some of them are not “bullet proved.” but they are very reliable in practice.

1. If all cash flows are positive, the IRR equation has a single solution.
2. If no preliminary estimation of the rate of return is available, the best option would be to find an approximate value using quadratic approximation. Then, the solution of IRR equation that is the closest is the right one.

3. If the rate of return is expected to be positive or slightly negative, which can be estimated on the basis of (15), then one should find the approximate value of rate of return using one of the following methods: Quadratic approximation, Linear approximation or Modified Dietz formula. Then, the closest solution of IRR equation is the right one.
4. If the sum of all absolute values of cash flows is of the order of the beginning market value or less, and the amounts of cash transactions are commensurate and cash transactions are distributed evenly through the investment period, then the IRR equation almost surely has one solution.

CONCLUSION

The inferences of this study are as follows.

1. IRR equation has a maximum possible number of solutions defined by the number of consecutive sign changes of its terms written in the descending order of powers. This result is valid for all forms of IRR equation including the case of real positive powers.

2. The business specifics of IRR equation restrict the maximum number of solutions. These specific features include the following:

- The sign of cash flows defines the number of possible real solutions. (Deposits are positive and withdrawals are negative.)
- The size and the sign of cash flows, including the beginning market value, and the time when cash transactions are made define the actual number of solutions of IRR equation.
- Rate of return cannot be less than minus one, that is $R \geq -1$;
- The ending market value of investment portfolio cannot be negative;
- The beginning market value of investment portfolio cannot be negative.
- The smaller cash transactions are, and the smaller the absolute value of their total is (total is found by summing up deposits and withdrawals with different algebraic signs) compared to the beginning market value of portfolio, the less solutions IRR equations has.

3. In some instances, the specifics of IRR equation is of such a limiting nature that it allows to determine the domain of interest rate which contains a unique solution. Such is the case of the mortgage IRR equation.

4. The non-uniformity of an actual investment growth leads to the situation that the intermediate market value of a portfolio can be negative when we use the value of rate of return calculated for the whole period. So, despite the obvious wisdom that in practice it should not be the case, we should accept this fact and do not impose such a condition in the problem of calculating rate of return based on IRR equation for the investment period with fixed length. On the other hand, when doing modeling of investment portfolios, this condition has to be taken into account, because the possible negativity of intermediate market values of portfolios is related only to calculation of rate of return for the whole portfolio, when the non-uniformity of investment growth cannot be accounted. In the modeling scenarios, we solve a *direct* problem, so that the *whole* investment process can be modeled, including the non-uniform change of rate of return throughout the whole period.

5. In the overwhelming majority of theoretically possible investment scenarios, and practically in all real situations, we can unambiguously locate the solution of IRR equation that makes sense from the business perspective by using appropriate approximations of IRR equation and some reasonable considerations.

REFERENCES

Chestopalov I., Chestopalov A. “The role of conceptual context in the problem of finding rate of return”, *The Journal of Performance Measurement* , 2008, Winter, Vol. 13, pp. 37-50.

Eschenbach, Ted., Baker IV, Elisha; Whittaker, John, “Characterizing the Real Roots for P, A, and F with Applications to Environmental Remediation and Home Buying Problems ” *The Engineering Economist*, Volume 52, Issue 1, January 2007, pages 41 – 65.

Shestopaloff, Yu. K. “Science of Inexact Mathematics. Investment performance measurement. Mortgages and annuities. Computing algorithms. Attribution. Risk valuation”, 2009, AKVY Press, 592 pp.

Shestopaloff, Yu. K. “Mortgages and Annuities: An Introductions”, 2009, AKVY Press, 205 p.

Shestopaloff, Yu. K. “Properties and Interrelationships of Polynomial, Exponential, Logarithmic and Power Functions with Applications to Modeling Natural Phenomena”, 2010, AKVY Press, 228 p.

Shestopaloff, Yu. K. “Mortgages and Annuities: Mathematical Foundations and Computational Algorithms”, 2010, AKVY Press, 287 p.

Shestopaloff, Yu. K. “Properties, relationships and solutions of equations composed of some elementary functions”, *Journal of Computational Mathematics and Mathematical Physics*, 2011, No. 5.