

NEW HIGH PERFORMANCE COMPUTATIONAL METHODS FOR MORTGAGES AND ANNUITIES

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Generally, mortgage and annuity equations do not have analytical solutions for unknown interest rate, which has to be found using numerical methods. Another issue is that these equations have multiple solutions. We discovered an interesting property of mortgage and annuity functions that can be beneficially used for computing the interest rate. Namely, these functions have a single minimum and one or no inflection point, while the value of interest rate that corresponds to the minimum is equal to approximately one-half of the value of the correct solution. The appropriate values of interest rates for the minimum and inflection point have a very simple analytical presentation. Based on this discovered property, we introduce new computational methods that allow unambiguously choosing the correct solution and finding a very accurate value of unknown interest rate without solving the mortgage or annuity equation at all. This value can be used directly or, if its accuracy is not sufficient, as the first initial value in iterative procedures. Such an initial value guarantees convergence to the correct solution. We also propose some additional computational enhancements, which, taken together, significantly improve the overall computational performance of mortgage and annuity related calculations.

Introduction

In this article, we propose new methods for computing interest rates for mortgages and annuities. Some may argue that modern computers are powerful enough, and system designers should not be concerned about computational efficiency. Even if computations take ten times more, would it make any difference if we compute an interest rate in 10^{-6} sec, or 10^{-5} sec? In fact, things are not so simple. Beside the one time computation of an interest rate, there is a large area of optimization problems, many of which require the computation of different combinations of parameters. For instance, many simulation models that are based on the Monte Carlo simulation method use very computationally intensive simulation procedures. So, if multiple computations of an interest rate are required (which often happens), then the computational efficiency of every repeated operation becomes important, and in many instances critical.

The proposed methods can be used in any relevant financial software application. Also, the high computational efficiency of these algorithms, which require small computational resources, makes their application in financial calculators very promising. The computational algorithms that we discuss in this article are based on an interesting discovered property of the mortgage (and annuity) IRR function, which allows efficient computing of interest rate.

The computational performance of numerical algorithms depends on two factors. The first one is the quality of a computational algorithm, such as, for instance, the fast convergence of Newton-Raphson's iterative algorithm. The second factor is the initial value used in the iterative process. If the initial value is chosen far from the solution, then

the iterative process requires more iterations, which slows down computations. In many instances, an inadequate initial value does not allow for the iterative procedure to converge to a solution.

In general, optimal criteria for choosing an initial value depend on the algorithm. Examples can be found in (Burden, 2005), chapter 10. However, when it comes to a specific problem, then one can substantially improve the accuracy of the initial approximation, and consequently the efficiency of the overall computational process, by taking into account problem specifics. In case of mortgage and annuities calculations, specific restrictions are imposed by the possible range of interest rates and number of payment periods. For instance, the annual interest rate for home mortgages is unlikely to be several tens or hundreds of percent. It usually lies within the range of several percent. So, in this case, it makes sense to choose an initial approximation of several percent.

In the financial industry, the software applications that serve numerical mortgage calculations, often use the initial interest rate of C/P . Here, C is a regular mortgage payment for a certain period, and P is the mortgage principal. This value can be obtained directly from the mortgage equation if one assumes that C compared to other equation's terms. This approach is often implemented in financial calculators. For brevity, we will call this method "CP-approximation". This method produces reasonable initial accuracy when the number of payment periods is large (over one hundred) and the interest rate is high (for an annual interest rate of roughly 15% or more). When the number of periods is less or the value of interest rate is smaller, then the iterative process that uses CP-approximation quickly becomes inefficient. Such calculations have to use a relatively large number of iterations (five and more) in order to obtain the required accuracy (usually, about 10^{-15}). There are also some other custom approaches that heavily rely on problem specifics and cannot be used outside their areas of application.

So, the choice of a more precise initial value of interest rate, which would reduce the number of required iterations, is important. The presented article addresses this issue and introduces general algorithms for finding the initial value of interest rate that can be used in any financial application that does mortgage or annuity calculations.

1. IRR equation for mortgages and annuities

We can transform the general form of IRR equation for mortgages, presented in (Shestopaloff, 2009, 2010), as follows, in order to make it more convenient for computational purposes.

$$P(1 + R)^{n+1} - (P + C)(1 + R)^n + C = 0 \tag{1}$$

where C is the mortgage payment; P is the principal (the borrowed amount); R is the interest rate (applied to one period); n is the number of payments.

This formula is similar in appearance to the IRR equation. We will call the left side of this equation the *mortgage IRR function*, and define it as follows.

$$F(R) = P(1+R)^{n+1} - (P+C)(1+R)^n + C \quad (2)$$

Formula (1) is essentially the IRR equation rewritten for initial conditions imposed by a mortgage as a specific lending instrument. (Other representations are possible, see Broverman, 1991.) These conditions include regular equal payments at equal intervals, until the principal is paid off. Otherwise, this is the same familiar IRR equation.

The CP-approximation can be derived from (1) as follows. Let us assume that

$$P(1+R)^{n+1} - (P+C)(1+R)^n \gg C$$

(Here, the notation " \gg " denotes a mathematical symbol "much greater".) The formula can be rewritten as $(1+R)^n(PR - C) \gg C$. We can see that this inequality holds true if the interest rate is high and the principal P is large compared to the regular payment C , which happens when the number of periods is large. If these conditions are fulfilled, we can obtain the equation $P(1+R)^{n+1} - (P+C)(1+R)^n \approx 0$, whose solution produces CP-approximation. For instance, when this approximation was used as the initial value for computing interest rate by Newton-Raphson's procedure, convergence was attained in a maximum of 3 iterations for the range of annual interest rates of 15 % or more, for a number of periods greater than two hundred.

Similarly to the mortgage function, we can derive the *annuity IRR function*, which has the following form for an ordinary annuity.

$$F_a(R) = P(1+R)^{n+1} - (P-C)(1+R)^n - C$$

The only difference between these functions is that the sign of a periodic payment has changed. Below, we consider calculation of interest rate for mortgages. However, exactly the same approach can be used for computing the interest rate for annuities. All formulas which we derive for mortgages are valid for annuities as well, if we substitute C by $(-C)$.

Equation (2) is a particular form of a regular IRR equation. So, it can be solved using the same methods. For instance, Newton-Raphson's iterative procedure, which can be derived as follows. Let us denote $x = 1 + R$. Then

$$x_{k+1} = x_k - \frac{Px_k^{n+1} - (P+C)x_k^n + C}{(1+n)Px_k^n - n(P+C)x_k^{n-1}} \quad (3)$$

After completing the computations, we can find the interest rate as $R = x - 1$.

We can also use an alternative approach, which can be introduced through the following substitution. Let us denote $y = (1+R)^{n+1}$. Then, a similar equation for Newton-Raphson's iterative procedure will be as follows.

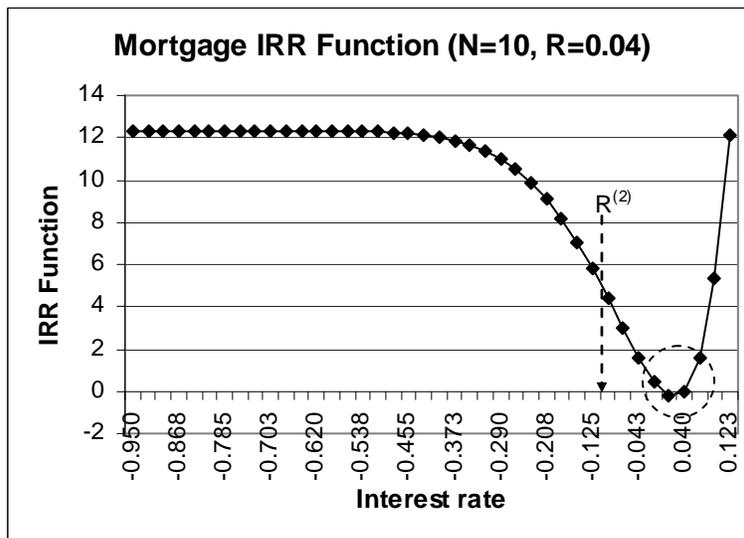
$$y_{k+1} = y_k - \frac{Py_k - (P+C)y_k^{\frac{n}{n+1}} + C}{P - \frac{n}{n+1}(P+C)y_k^{\frac{n}{n+1}}} \quad (4)$$

Once the iterative procedure is complete, the interest rate can be found as

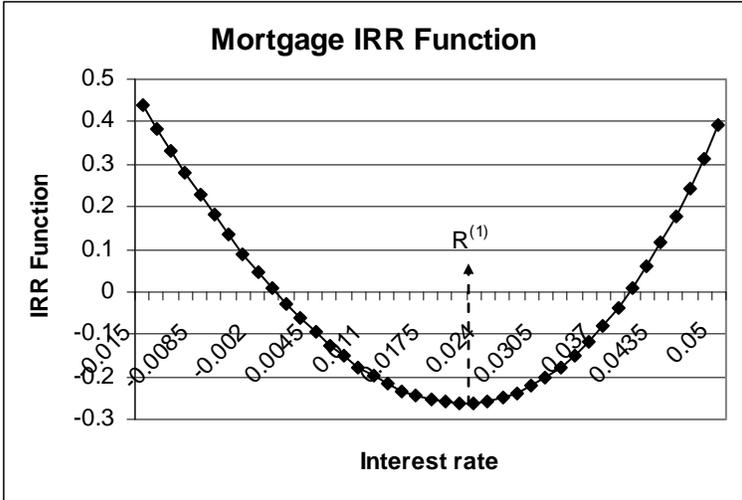
$$R = y^{\frac{1}{n+1}} - 1$$

2. Specific properties of the mortgage IRR function and its characteristic points

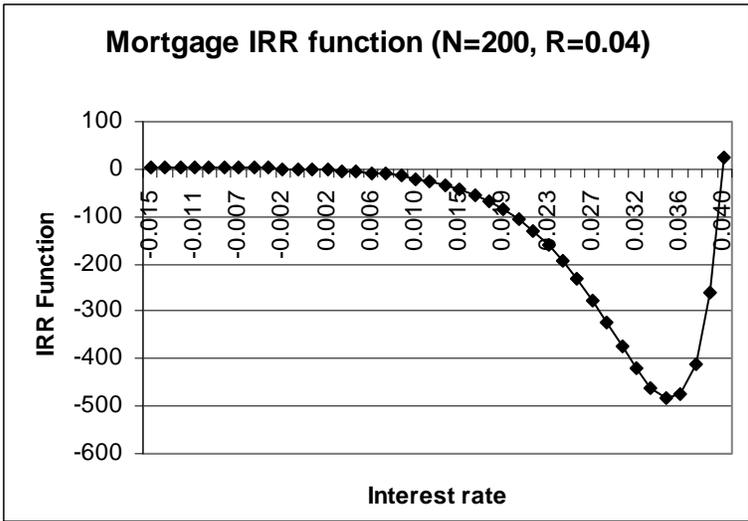
The mortgage IRR function has some interesting properties that make it possible to compute interest rates much more efficiently than the general form of IRR equation allows. We consider these properties below. We need them in order to find approximate values of interest rate that can be used by themselves or as the initial value for iterative algorithms.



a



b



c

Fig. 1. Graph of the mortgage IRR function. a – the overall view of function when the number of periods $N=10$; b – the same graph enlarged below the abscissa (circled area); c – the graph for a larger number of periods ($N=200$).

Let us take a look at the graph at Fig. 1 that represents the mortgage IRR function. Note the area in the circle where the IRR function’s curve “sinks” below the abscissa. These two intersections of the abscissa by the graph represent solutions of the mortgage IRR equation.

The first solution is $R = 0$. The existence of this solution can be proved by substituting this value into (1) and doing the appropriate transformation as follows: $P - (P + C) + C = 0$. So, the value of $R = 0$ is a solution for every combination of possible input parameters P and C . The second solution is the one we are looking for.

Note that the graphs of the IRR function are skewed to the right, so that the minimum point is located slightly to the right of the middle point between the two solutions. We will provide more detail with regard to this property of the IRR function later. At this point, it is important to note that the double distance between the minimum of the IRR function and its first solution $R=0$ is close to the value of the second solution, which is the solution we need. This value slightly exceeds the real solution when the number of payment periods is small. Note that when the number of periods increases, then the divergence of this double distance from the real solution increases too.

3. Asymptotic behavior of the mortgage IRR function. Characteristic points

Let us consider other properties of the mortgage IRR function. To the right of the second solution, the graph of the mortgage IRR function monotonically (continuously) increases to infinity, and never intersects the abscissa again. This behavior can be proved by considering (1) for large values of R . In this case, we can neglect C , and the first term monotonically overtakes the second term, thus forcing the IRR function to go to infinity.

$$\begin{aligned}
 & \lim_{R \rightarrow \infty} P(1+R)^{n+1} - (P+C)(1+R)^n + C = \\
 & = \lim_{R \rightarrow \infty} (1+R)^n (P(1+R) - P - C) = \\
 & = \lim_{R \rightarrow \infty} (1+R)^n (PR - C) = \infty
 \end{aligned} \tag{5}$$

Alternatively, we can explore the first derivative of the mortgage IRR function. It is defined as follows.

$$F^{(1)}(R) = P(n+1)(1+R)^n - n(P+C)(1+R)^{n-1} \tag{6}$$

Equating the first derivative to zero, we will find the following value.

$$R_0^{(1)} = \frac{nC - P}{(n+1)P} \tag{7}$$

We know that the point where the first derivative becomes zero is an extremum, which can be either a maximum or a minimum. In our case, based on the graph in Fig. 1, we can assume that this is a minimum. We can prove this more rigorously using the second derivative, which should be positive (as function is convex at its minimum point). The second derivative can be found as follows.

$$F^{(2)}(R) = n(n+1)P(1+R)^{n-1} - n(n-1)(P+C)(1+R)^{n-2} \tag{8}$$

If we substitute the value of interest rate from (7) into this formula, when the first derivative is equal to zero, then we will find that $F^{(2)}(R_0^{(1)}) = n(1+R_0^{(1)})^{n-2}(P+C) > 0$.

So, the mortgage IRR function has one extremum, which is a minimum, because the function is convex at this point. (Certainly, the same assertion is true for the annuity IRR function.) We can find from (6) that the first derivative is negative to the left of this minimum, which means that the function monotonically decreases in this domain, while the first derivative is positive to the right of this minimum. This means that the function grows monotonically in this domain.

We will need the value of interest rate when the second derivative of IRR function becomes zero. In geometrical terms, this value corresponds to function's inflection point. Equating the second derivative (8) to zero, and solving this equation, we will find the following value.

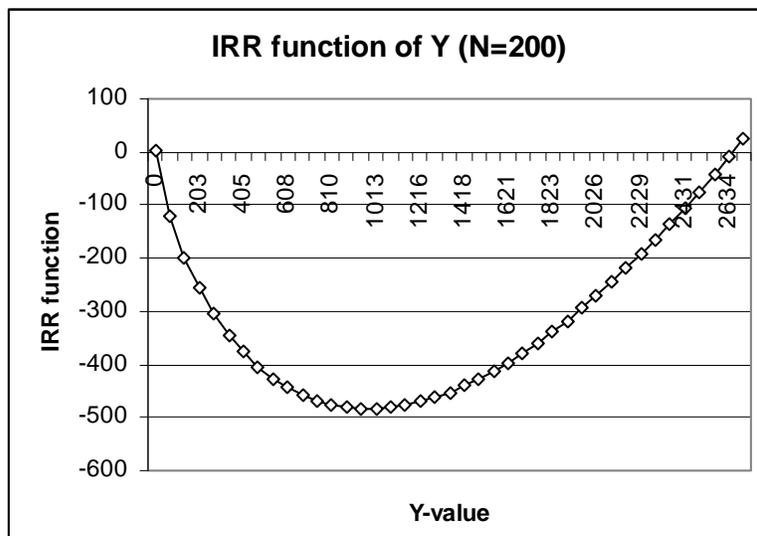
$$R_0^{(2)} = \frac{C(n-1) - 2P}{(n+1)P} \tag{9}$$

4. Transforming the mortgage IRR equation

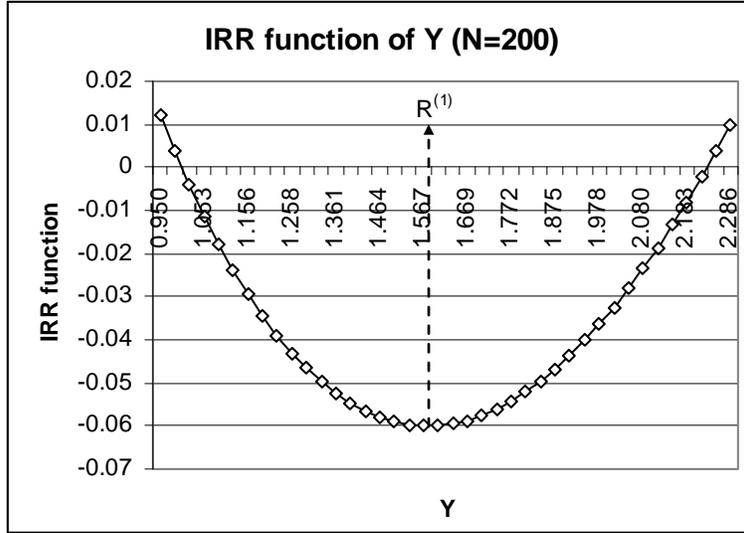
Below, we continue to explore specific properties of mortgage IRR functions, this time analyzing transformations of the original equation (1).

Let us consider the graphs of the IRR function as it depends on the variable y , which is defined as $y = (1 + R)^{n+1}$ (we used this substitution in formula 3). The mortgage IRR function, in this case, is as follows.

$$F(y) = Py - (P + C)y^{\frac{n}{n+1}} + C \tag{10}$$



a



b

Fig. 2. Graphs of mortgage IRR function as a function of Y . a) $R_0 = 0.04$; b) $R_0 = 0.004$.

In general, we can find the maximum number of abscissa intersection functions (2) and (10) (the number of solutions of the appropriate equations) using the analog of Descartes Rule of Signs for generalized polynomials. Application of this rule produces a maximum number of two solutions. Formulation of this rule and proof of appropriate theorems can be found in (Shestopaloff, 2010, 2011) and formulation in (Rahman, 2002).

5. Approximate solution of IRR equation

We will introduce methods to obtain approximate solutions based on specific properties of the mortgage IRR function, which we discovered. On the other hand, despite the practicality of these methods, they are not our ultimate goal. We consider these methods in order to facilitate the introduction of a more advanced method, based on similar considerations.

Fig. 2 shows graphs of the mortgage IRR functions for different parameters in case of Y -substitution (10). We can see the same two solutions, as it was the case with the IRR function (2). One solution is $y = 1$ (which corresponds to $R = 0$), and the other solution is some $y > 1$, which is the solution we need. We can also see that unlike in the case of Fig. 1, when the graphs were skewed to the right, the graphs in Fig. 2 are skewed *to the left*, so that the minimum is located to the left of the middle point between the solutions. This trend is more prominent when the interest rate increases. This is what we can see in Fig. 2, a. The graph in Fig. 2, b for a smaller interest rate is almost symmetrical, so that the point of minimum $R_0^{(1)}$ is close to the midpoint between the solutions of the mortgage IRR equation.

So, in the case of Y -substitution we discovered that the double distance between the first solution $R=0$, and the point of minimum of IRR function, is close to the value of the

second solution. In addition, because of the overall skew of the IRR function to the left, this value is slightly less than the second solution.

The real solution can therefore be approximated by a double distance between the point $R=0$ and the point of minimum $R_0^{(1)}$, that is by $2R_0^{(1)}$. However, when the number of periods is large - this scenario is represented by Fig. 1, c - the more correct approximation is when instead of $R=0$ we use the value of R that corresponds to the inflection point.

Below, we will refer to the aforementioned approaches as X-approximation (which we found on the basis of equation 1), and Y-approximation (based on equation 10). Let us find the value of X-approximation first, using the considerations presented above.

$$\begin{aligned}
 R_X &= 2(R_0^{(1)} - R_0^{(2)}) + R_0^{(2)} = \\
 &= 2 \frac{(nC - P)}{(n+1)P} - \left(\frac{C(n-1)}{P(n+1)} - \frac{2}{n+1} \right) = \frac{C}{P} \quad \text{if } R_0^{(2)} > 0 \\
 R_X &= 2R_0^{(1)} = \frac{2(nC - P)}{(n+1)P} \quad \text{if } R_0^{(2)} \leq 0
 \end{aligned} \tag{11}$$

Note that the first formula is the approximate solution for a large number of payment periods, and / or for the relatively large interest rates. We obtained it earlier by analyzing the original mortgage IRR equation (1).

Let us find the Y-approximation. In order to do that, we have to know the point of minimum $y_0^{(1)}$ of IRR function $F(y)$. The first derivative of this function is as follows.

$$F^{(1)}(y) = P - \left(\frac{n}{n+1} \right) (P + C) y^{-\frac{1}{n+1}} \tag{12}$$

Equating (12) to zero, we will find the value of $y_0^{(1)}$ that corresponds to the minimum of function $F(y)$.

$$y_0^{(1)} = \left[\left(1 + \frac{C}{P} \right) \left(\frac{n}{n+1} \right) \right]^{n+1} \tag{13}$$

The second derivative of function $F^{(1)}(y)$ is never equal to zero, which can be easily proved by differentiating (12). So, we can use the point $y = 1$, which corresponds to the value $R = 0$, as the first reference point. Hence, the value of Y-approximation can be found as follows.

$$Y_0 = 1 + 2 \times \left[\left[\frac{n(P + C)}{P(n+1)} \right]^{n+1} - 1 \right] \tag{14}$$

The corresponding interest rate R_Y can be found from (14).

$$R_Y = Y_0^{\frac{1}{n+1}} - 1 \quad (15)$$

The introduced X and Y approximations can therefore be used as initial values for iterative procedures, such as the Newton-Raphson's method, or quadratic approximation (Shestopaloff, 2009). However, these values have another useful application. Any initial approximation to the left of the minimum will result in solution $R = 0$, or no solution at all. The initial approximation has to be chosen to the right of the minimum of the IRR function, in order to guarantee the convergence of the iterative procedure. So, the point of minimum $R_0^{(1)}$, defined by (7), and the point of minimum (13), should be used when choosing the initial value for iterative procedures for finding interest rates. In particular, for equation (1), this condition is stated as follows: $R_0 > R_0^{(1)}$.

6. The accuracy of X and Y approximations, and their computational performance

We saw that the choice of the first approximation is an important issue which determines the convergence or divergence of the iterative procedure and computational performance of iterative algorithms in general. We should choose the first approximation as close to the solution as possible, in order to implement a computationally efficient iterative procedure. Let us now analyze the accuracy of X and Y approximations and computational performance when they are used with Newton-Raphson's iterative procedure.

Table 1 illustrates the accuracy of X approximation. The absolute approximation error is measured in percent of interest rate. The relative approximation error is defined as the absolute approximation error divided by the value of interest rate. In the practical range of interest rates, up to 55%, the approximation error is a few percent. The number of required iterations for convergence is 2-3. The number of payment periods ranges from 30 to 100.

We can see that when the required accuracy is on the order of 10^{-4} , and the value of unknown interest rate does not exceed 4–5%, then X-approximation can be used directly for finding interest rates. For instance, when the interest rate is 3.7%, the absolute error is 2×10^{-4} .

Table 1. X-approximation errors, depending on the interest rate for the whole period.

Interest rate, %	Relative Approximation error, %	Absolute Approximation error, %	Number of iterations
3.73	0.5	0.02	2

28	3.6	1.0	3
55.5	6.7	3.7	3
87	9.4	8.2	4
122	11.5	14	4
161	14.3	23	4
206	16.1	34	4
255	18.8	48	4
310	20.6	64	5
371	22.6	84	5

Table 2 presents similar data for Y-approximation. Note that the approximation value is less than the actual solution, because of slower pace of growth of the corresponding IRR function.

When the interest rate does not exceed 30 %, computational performance of both approximation algorithms is very impressive. Summary of performance results for this range of interest rates is presented in Table 3.

Overall, the presented results show that both approximations may, in some instances, have practical value for solving the mortgage IRR equation.

Table 2. Y-approximation error, depending on the interest rate for the whole period; the predefined accuracy is 10^{-10} .

Interest rate, %	Relative approximation error, %	Number of iterations
3.73	-0.36	2
28	-2.45	3
55.5	-3.80	3
87	-4.46	3
122	-8.21	4
161	-9.97	4
206	-11.7	4
255	-13.3	4
310	-14.8	4
371	-13.0	4

Table 3. Required time to calculate 1,000,000 interest rates on a desktop computer, depending on the computational algorithms; $R \leq 30\%$.

Number of periods	X-approximation, sec	Y-approximation, sec
12	0.73	1.02

120	1.00	1.12
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7. Generalization of X- and Y- approximations. A-approximation method

The idea of the method that we present in this section is based on the observation that when we decrease the power of the first term in the mortgage IRR equation, the graph of the function becomes skewed more to the left, which means the minimum is displaced to the left, and vice versa. Let us consider the general form of this equation.

$$F(x) = Px^A - (P + C)x^{\frac{n}{n+1}} + C \quad (16)$$

Note that when $x = 1$, we have $F(1) = 0$ and $x = 1$ is the first solution of equation $F(x) = 0$.

In general, when we increase the value of A , the minimum of function (16) moves to the right, so that there is always a value of A such that the function's minimum is located exactly in the middle of the interval between the two solutions. Of course, such a value of A also depends on other parameters, in particular, on the principal amount and period payments. However, it turns out that dependencies on the last parameters are not critical, and, considered as a function of the number of periods, A can be used to provide a much better approximation to the interest rate than X and Y approximations.

So, we have to find the minimum and inflection point of function (16). This can be done in the same way as before. That is we have to find the first and second derivatives, equate them to zero, and solve the obtained equations with respect to x . Let us find the first derivative.

$$F^{(1)}(x) = APx^{A-1} - A\frac{n}{n+1}(P + C)x^{\frac{n}{n+1}-1} \quad (17)$$

Equating the right side of (17) to zero, and solving this equation with respect to x , we will eventually obtain the following.

$$x_0^{(1)} = \left[\left(1 + \frac{C}{P} \right) \left(\frac{n}{n+1} \right) \right]^{\frac{n+1}{A}} \quad (18)$$

Formula (18) can be verified as follows. In case when $A=1$, we have to obtain the point of minimum for Y-approximation, result (13). Equation (18) satisfies this requirement. When we substitute $A=1$, we obtain (13), which is the minimum of IRR function for the Y-approximation. In the same way, when we substitute $A=n+1$, then we have $x = R + 1$, thus (18) can be transformed into (7), which represents the minimum for the X-approximation.

The next step is to find the value of x that corresponds to inflection point. As we discovered previously, (18) may not necessarily have an inflection point, as it was the case for Y-approximation. So, when implementing the A-approximation, one should check for the existence of this point. If the point does not exist, the value of $x=1$ should be used instead.

The second derivative can be found as follows.

$$F^{(2)}(x) = A(A-1)Px^{A-2} - A\left(A\frac{n}{n+1} - 1\right)\left(\frac{n}{n+1}\right)(P+C)x^{\frac{A-n}{n+1}-2} \quad (19)$$

Solving equation $F^{(2)}(x) = 0$, we find the value of $x_0^{(2)}$ that corresponds to the inflection point.

$$x_0^{(2)} = \left[\left(\frac{1}{A-1} \right) \left(1 + \frac{C}{P} \right) \left(\frac{n}{n+1} \right) \left(\frac{An-n-1}{n+1} \right) \right]^{\frac{n+1}{A}} \text{ if } x_0^{(2)} > 1 \quad (20)$$

If $x_0^{(2)} \leq 1$, then we should assume that $x_0^{(2)} = 1$. We can see from (20) that inflection point exists if $A \geq 1 + \frac{1}{n}$.

Finally, we can write the approximation formula for the value of x_A , when the function $F(x)$ intersects abscissa for the second time. In other words, we can find the second solution of equation $F(x) = 0$ using the same approach as for computing the X-approximation. Namely, if the inflection point of IRR function is greater than one, then we have to compute the double distance between the minimum and inflection point, and add this value to the x that corresponds to the inflection point. If this value is less than one, then we substitute it by 1. The following formula thus covers all scenarios.

$$x_A = x_0^{(2)} + 2(x_0^{(1)} - x_0^{(2)}) \quad (21)$$

Once the approximate value of x_A is determined, we can find the approximate value of the interest rate from (21) as follows.

$$R_A = (x_A)^{\frac{A}{n+1}} - 1 \quad (22)$$

8. Adding CP-approximation

In addition, when the number of periods is large, we will complement the A-approximation method with the earlier discussed CP-approximation. We proved that in this case, we can approximate the value of interest rate as

$$R_{CP} = \frac{C}{P} \quad (23)$$

Substituting this into (22), we get

$$x_A = 1 + \frac{C}{P} \quad (24)$$

when we have

$$A = n + 1 \quad (25)$$

For this scenario, formula (16) effectively transforms into the left side of the original IRR equation (1), which validates the transformation. So, we can use the CP-approximation for (16) as well.

The question is: how we can find the threshold which will work as a trigger which switches the computations from A-approximation to CP-approximation. It turns out that the value of $\frac{C}{P}$ itself can be used for this purpose. We will discuss this in the subsection devoted to the algorithm's implementation.

Overall, both A- and CP approximations are not technically complicated. What makes them a little difficult to understand is the underlying unconventional idea, which is, in fact, is simple and elegant.

9. Practical implementation of A-approximation method. Accuracy evaluation. Numerical examples

We need the following components in order to implement the method.

1. A threshold function, which is the trigger between A-approximation and CP-approximation.
2. A "power function" that computes the value of A .

Let us begin with the threshold function. First, we have to define the domain of interest rates to which we would like to apply the approximation. In our numerical example, we specified a domain of up to 100 % annual interest rate. Note that we can implement different variations of the A-approximation. For instance, the range of possible numbers of periods can be divided into several sub-ranges, and additional trigger functions that call appropriate A-approximations for particular numbers of periods can be implemented. This way, the approximation accuracy can be further increased by orders of

magnitude. For instance, if the range of numbers of periods [2, 100] is subdivided into four intervals, [2-6], [6-18], [18-40] and [40-100], then the accuracy increases, on average, by 23 times.

10. Threshold function

In our case, the threshold function $T\left(n, \frac{C}{P}\right)$ is very simple and triggers only when $n > 100$. The graph of the value of the relative approximation accuracy, depending on the number of periods, is shown in Fig. 3.

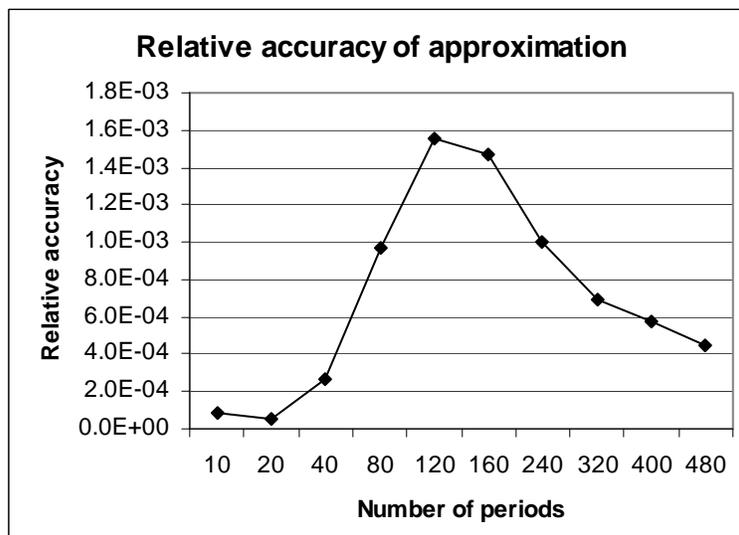


Fig. 3. Relative accuracy of combined A and CP approximations for the range of interest rates up to 100 %.

The form of the graph in Fig. 3 is well explained by the previous considerations. We know that when the number of periods is relatively small, the graph of the mortgage IRR function is above the abscissa, so that two solutions are close to each other, and the function’s minimum is located close to the middle point between two solutions. We can expect that the approximation accuracy should be high, and this is what the graph in Fig. 3 confirms. With the increase of the number of periods, the graph of IRR function “sinks” more below the abscissa, the distance between solutions increases, and even minor variations of the power A should noticeably impact the accuracy. This is why the approximation error increases. This trend eventually is intercepted by the CP-approximation, when it becomes more accurate than A-approximation.

Note that the overall achieved approximation accuracy is very good. However, if we reduce the interest rates’ range to annual 10 %, then the accuracy increases two-three orders of magnitude, which in many cases is already sufficient accuracy for practical applications. So, subdividing the range of interest rates into several sub-domains is a

beneficial approach, while it adds very small computational overhead. We can find the initial value of interest rate, in order to determine the sub-domain, using X-approximation (for smaller number of periods) or Y approximation (for larger number of periods).

11. Power function

The “Power function” $A = A(n)$ is also very simple function that depends only on the number of periods. In our case, we subdivided the domain of period numbers into two. One is for $n = \{2,20\}$, and the other one for $n = \{20,800\}$, so that $A(n)$ is the composite of two simple functions.

$$A(n) = \frac{1.2}{n^{0.925}}, \text{ for } n = \{2,20\}; \quad A(n) = \frac{1.48}{n}, \text{ for } n = \{20,800\};$$

The approach, which we used to find the representation of $A(n)$, was based on minimization of relative error for the whole range of interest rates.

12. Iterative algorithms

The next step is incorporation of the approximation procedure into iterative algorithms. We used two iterative methods. One is the Newton-Raphson’s iterative algorithm, and the other one is the quadratic iteration from (Shestopaloff, 2009). If we rewrite formulas for quadratic approximation, then they will transform as follows for the mortgage IRR function.

$$\begin{aligned} a &= \frac{1}{2} n x_k^{n-2} [x_k P(n+1) - (C + P)(1-n)] \\ b &= x_k^{n-1} [x_k P(n+1) - n(P + C)] \\ c &= x_k^n [x_k P - (P + C)] + C \end{aligned} \tag{26}$$

where $k = 0,1,2,..$ and $x_0 = x_A$.

Then, the iteration algorithm is represented as a solution of quadratic equation, plus the previous iteration.

$$x_{k+1} = x_k + \frac{-b + \sqrt{b^2 - 4ac}}{2a} \tag{27}$$

It turned out that this quadratic iteration algorithm is computationally superior to Newton-Raphson's method because of the smaller number of iterations. We evaluated the computational performance of both methods in the range of annual rates up to 100 % and the number of periods from 2 to 800. Table 4 presents the results.

Table 4. Computational efficiency of quadratic and Newton-Raphson's iterations for the combined A-approximation, and CP-approximation only for the number of periods [2-120]. Time required to calculate 1,000,000 interest rates on a desktop computer, in sec.

	Quadratic iteration A-approximation	Newton-Raphson
A-approximation	0.72 – 0.91	1.06 – 1.37
CP-approximation	3.12 – 5.64	3.42 – 4.55

The A-approximation provides the most computational gain, compared to CP-approximation, in the range of periods from 2 to 120. CP-approximation alone works well for the number of periods above 200. We can see from this table that A-approximation provides very good and stable computational performance in the whole range of practical values of interest rates.

We also researched the feasibility of using the proposed A-approximation method in financial calculators, considering the base model HP 20b. The very preliminary evaluations showed that the performance gain is of the order of 1.7-2.5 times, due to reduced number of iterations.

Fig. 4 shows the worst scenario that we encountered with regard to the number of iterations in case of quadratic iteration, while the usual number of iterations does not exceed two.

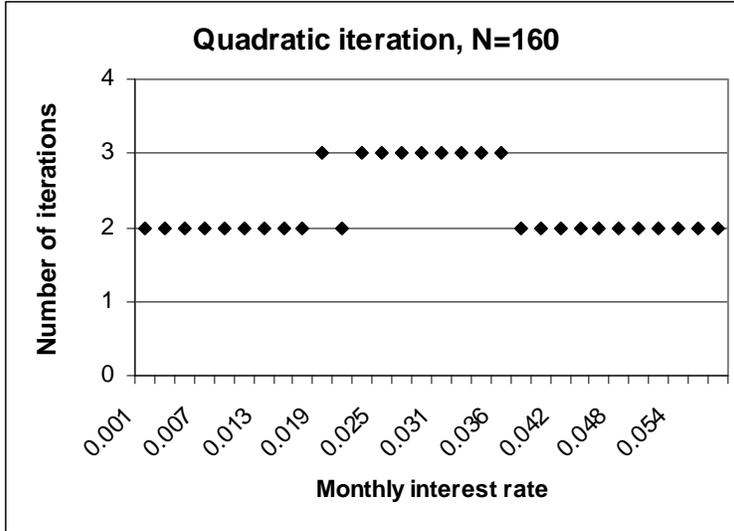


Fig. 4. Number of quadratic iterations depending on the monthly interest rate. The number of payment periods is 160, iteration accuracy is equal to 10^{-13} .

The number of iterations for the Newton-Raphson's method is usually in the range 2-3, very rarely increasing to four iterations. The implementation of Newton-Raphson's method in case of A-approximation is based on formula (16). The interest rate is calculated in the very final phase from the value x obtained in the last iteration (22). This is why the iteration procedure in case of A-approximation deals only with the value of x . With this regard, we can call it as A-iterative algorithm. Using (16) and (17), we can write it as follows.

$$x_{k+1} = x_k - \frac{F(x_k)}{F^{(1)}(x_k)} \quad (28)$$

Overall, the computational performance achieved by A-approximation method (combined with the CP-approximation) and A-iteration algorithm is very good. In our numerical illustrations the computational performance has been several times better than the best approximation (CP-approximation) that is used today (see Table 4). Such substantial improvement in computational time can have significant impact to the system design approaches and business applications, as well as to the design of financial calculators (this is the most computationally expensive calculation). The note especially appeals to information systems performing optimization of investment strategies, because such systems require exhaustive computations based on analysis of different permutations of influencing parameters. Such efficient algorithms are also indispensable computational tools for the systems that recalculate mortgages and other lending instruments or do simulations on a regular basis, which is a common practice in lending institutions.

13. Defining iteration accuracy

Previously, we discussed briefly the iteration accuracy. In particular, we discovered that the iteration accuracy in case of some iterative methods, such as Newton-Raphson's algorithm, does not guarantee that we found an interest rate with the same precision. Some iterative methods, such as, for instance, bisectional algorithm, are free from this drawback.

However, even with regard to iterative methods, which use the difference between successive iterations as a measure of accuracy, the issue is not straightforward. In this subsection, we will discuss some of these problems. Let us consider formula (22), which we reproduce below.

$$R_A = (x_A)^{\frac{A}{n+1}} - 1 \quad (29)$$

In fact, we are interested in the accuracy of value R_A , while A-iterative algorithm (28) produces value x_A . So, if we require that the final value has to be found with an accuracy a , then the value of x_A has to be calculated with the accuracy of $a \times \left(\frac{n+1}{A} \right)$, according to the rules for finding accuracy of derivative values. In case of A-iteration algorithm, this can be a large correction of the order of number of periods, which can save one-two iterations.

Another consideration is this. Do we need to calculate the interest rate for loans \$100 and \$10,000,000 with the same accuracy 10^{-15} , as some information systems, and even some handheld calculators, implement? The answer is no. In fact, even if the requirement is to round the total sum to cent, the actual required accuracy in these two cases can differ as much as 10^7 . The procedures how to compute the actual required accuracy for the iterative procedure, given the accuracy of the final evaluation, are well defined, and we will not elaborate on this topic. We just wanted to pay attention to this purely technical issue which is often neglected in the systems' implementations, while it can save 10-30% of precious time and computational resources by applying minimum efforts during the design and implementation phases.

Conclusion

The presented results show that iterative procedures are not the only option when we have to find the interest rate for financial instruments with regular payments, such as mortgages and annuities. In many instances, the introduced methods allow finding interest rates with high accuracy without using any iterative procedures at all. Depending on particular requirements, the methods can be further adjusted for higher accuracy and efficiency. The proposed algorithms improve the computational efficiency of mortgage and annuity related financial software applications from tens of percent to several times.

Such software applications can be deployed on all kinds of computers. Besides, the simplicity and high computational efficiency of introduced algorithms make them very useful for devices with low computational resources, such as handheld and desktop financial calculators.

We discovered a condition for the initial value that guarantees convergence of the iterative process to the right solution. This is an important computational restriction that makes the solution of the mortgage IRR equation unambiguous when one uses iterative algorithms, such as Newton-Raphson, or similar ones.

All obtained results are valid for annuities if we substitute the value of mortgage payment C by $(-C)$, assuming that C is an annuity payment. In fact, when the purchaser of an annuity *receives* payments, we can consider this scenario as a conventional mortgage, because the value of the annuity in the annuity equation in this case is negative, which transforms it to the mortgage IRR function (2).

We also discovered additional ways of improving the computational performance of financial software systems that deal with such mortgage and annuity related calculations. In particular, varying the accuracy of computations, which is defined by input parameters, can significantly reduce the computational workload.

Overall, this newly discovered approach for computing interest rates of mortgages and annuities opens a whole new area. It significantly improves the arsenal of efficient computational tools for analytical studies and routine business computations related to these financial instruments.

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