A MODEL FOR A GLOBAL INVESTMENT ATTRIBUTION ANALYSIS, BASED ON A SYMMETRICAL ARITHMETIC ATTRIBUTION MODEL

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Abstract

This article proposes a method for a global investment attribution. This method can be applied to equity portfolios, when portfolios are composed of assets traded in different currencies; active currency management strategies, and other investment policies. We consider the problem of computing and decomposition of rates of return for such investments portfolios with multiple currencies, and provide a detailed and complete study of mathematical aspects, illustrated by a numerical example confirming the validity of analytical results. In the attribution analysis, we used a symmetrical arithmetic attribution model to find the global attribution parameters. The advantage of this attribution model is that it does not have the ‘noise’ terms, such as an interaction; thus, it provides more objective analysis. In addition, the article considers currency hedging and multiple currency conversions with regard to the global attribution analysis and computing rates of return. We provide a mathematical model, adequately representing the currency hedging and currency conversion operations for the purposes of reporting and analytical studies.

1. Global Attribution

Analysis of the global investment attribution (or global attribution in short) relates to analytical studies of investments in the global markets, when currency exchange is involved. The portfolio’s rate of return in this case is composed of an investment risk premium, and the currency exchange return. We will use the following notations.

Base currency relates to investor's country.
Local currency relates to a country where the investment is made.

The following example below illustrates this situation. Suppose a European investor decided to invest in Australia. At first, he has to buy a local currency, which is Australian dollars, at a certain exchange rate using the euro as the base currency. Then he has to buy shares in the Australian market paying Australian dollars. At some point in the future he will sell the shares also in Australian dollars. At this point, he can calculate the rate of return in Australian dollars, or, in other words, in a local currency. Finally he has to convert Australian dollars into his base currency, which is euro. Now he can calculate the rate of return in the base currency.

Thus, the total return has two components. One is due to the change of the shares' price in Australian dollars, the other appears due to fluctuations of currency exchange rates. In the numerical example, we will start with one hundred euros and will follow through the series of steps depictured in Fig.1.

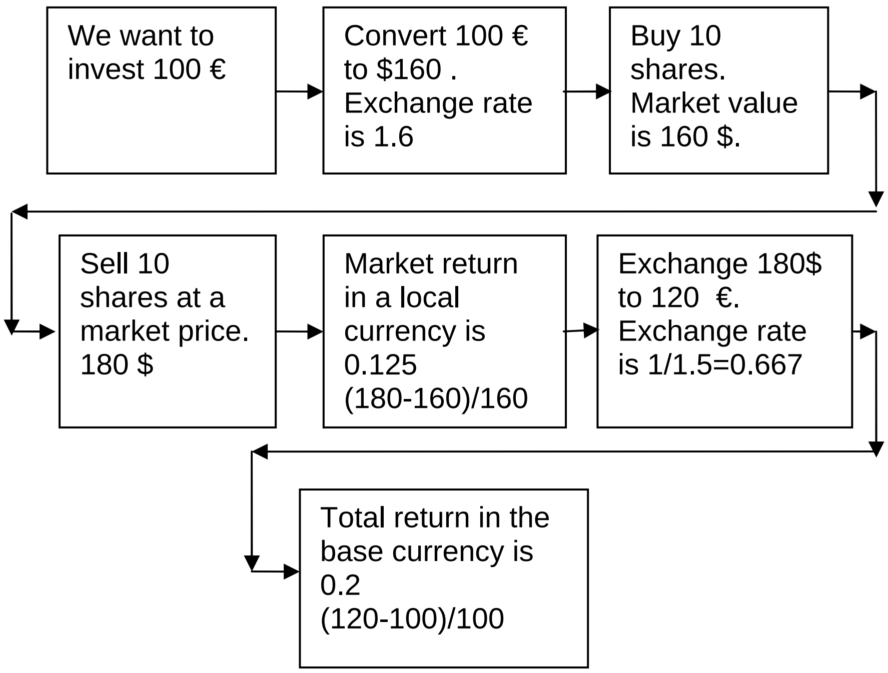

Fig. 1. Investing in the global markets.

So, while Australian investor receives 12.5 % return, his European counterpart enjoys 20 % due to the favorable currency exchange rates. A currency return is approximately 6.67 %.

Please note that if we sum up these two returns, then we will get 19.167 %, but not 20%. It happens for a reason, and we will discuss this issue later.

So, the investment attribution analysis should take into account the presence of two different factors influencing the resulting return. The idea is to decompose the total global return both for the portfolio and benchmark into two returns. One is a *market return* (that is either a portfolio market return, or a benchmark market return), and the second is a *currency return*. Then, we should compare the portfolio's market return to the benchmark's market return, and the portfolio's currency return to the benchmark's currency return. Each comparison is done exactly in the same way as it is usually done for an equity portfolio, when we invest only in a local market. These attribution methods for equity portfolios invested in domestic markets are well described in (Shestopaloff, 2009), (Spaulding, 2003).

We will use familiar attribution parameters such as the industry selection and stock selection for the market return (both for the portfolio and benchmark), and accordingly a country selection and a currency selection for the currency return. (We should make a note that the reader can encounter different terminology with regard to a currency attribution.) One of the known methods for a global attribution is Karnosky-Singer's method (Karnosky, 1994). However, the method we propose is independent on their model. We will derive it from scratch, so that no prior knowledge of any specific method will be required. However, it is good to know what has been done before. One of the advantages of the new method is its better objectivity, and the proof of this declaration will be another thread of this article. The approach that we propose does not include an interaction and other noise factors, which in many instances are incorporated implicitly into other models (except for some attribution parameters in case of geometric attribution models). Besides, the proposed method does not use approximations or any other assumptions to simplify the computational algorithms and thus jeopardizing the models' efficiency and accuracy. With this regard, tt is a *precise* algorithm.

2. Global Attribution Basics

As an example, let us find the total rate of return of an investment portfolio, given the business scenario described above. We will use the following notations. The beginning market value of an investment portfolio (in the base currency) that we would like to invest into foreign markets is *B*. The ending market value of this investment portfolio is *E*. The exchange rate for the conversion of the base currency into a local currency is e_f ("*f*" means "forward", in the numerical example above this is the value of 1.6 for the conversion of euro into Australian dollars). The exchange rate for the reverse conversion is e_b ("*b*" means "back", this is a value of 0.667 in the above example). Then, we can find the ending value of the investment portfolio as follows.

$$E = Be_f e_b (1 + R_M) \qquad (1)$$

where R_M is a portfolio's market return earned in a local currency.

Using this ending market value of the portfolio, we can find the total rate of return for this investment portfolio in the following form.

$$\begin{aligned} R_T &= \frac{E - B}{B} = \frac{Be_f e_b (1 + R_M) - B}{B} = \\ &= e_f e_b (1 + R_M) - 1 = (1 + R_C)(1 + R_M) - 1 \end{aligned} \qquad (2)$$

where R_T and R_C are the total portfolio's return and the portfolio's currency return accordingly.

Equation (2) represents a geometrical linking operation for the portfolio's market and currency returns. This is how the portfolio's total rate of return should be calculated when somebody invests into global markets, whe a currency exchange is involved. We can transform (2) further, obtaining the following compact formula.

$$R_T = R_M + R_C + R_M R_C \qquad (3)$$

Formula (3) explains why we did not obtain 20 % return in the numerical example, when we added the portfolio's market and currency returns. This is because the third term in (3) has been missed, that is the term $R_M R_C$. If we substitute the previously found values into this equation, we will get exactly 20 % of the total return. So, nothing is lost and each and every investors' penny is accounted properly.

Many publications define the total return for a global investment portfolio as a simple sum of market and currency returns. This is not the right approach, as the reader can see from

(3). However, this is an acceptable approach if the rates of return are really small. If we decide to use this simple additive approach, we should agree upon the required accuracy of the presentation, and we have to control this accuracy throughout all computations. Unfortunately, such an accuracy control is not always possible. In any case, these sorts of assumptions with regard to approximate nature of models should be set explicitly up front in order not to confuse the users of the methods. Regrettably, many papers lack such analysis of models' application domain.

The presence of the third term in (3) imposes some problems with regard to attribution analysis. How are we going to combine the attribution results done separately for the currency and market returns if they do not have an additive property? On the other hand, do we really need to combine these results? Even if we manage to do this, what would be the business sense of thus obtained attribution parameters? In fact, these parameters can be found directly in the base currency. So, from the business perspective, we do not see much need to develop the approach capable of such amalgamation of the market and currency attribution values. With this regard it means that both a currency attribution and a market attribution can be analyzed *separately* without jeopardizing the quality of investment attribution analysis. This approach will deliver *comprehensive*, *meaningful* and *correct* results.

3. Choosing the base attribution model

We will introduce a new global attribution model in this section. Our goal is to do this on the basis of arithmetic attribution model that does not have noise factors, such as an interaction in the BHB model (Brinson, 1986). A new model will provide more adequate and meaningful results because these noise factors will not present from the very beginning.

We will use an arithmetic attribution method introduced in (Shestopaloff, 2009), (Shestopaloff, 2008). This symmetrical attribution model does not have interaction terms or other terms considered as the model's noise factors. Attribution parameters of this model are defined as follows.

$$I = \frac{(r+R)}{2}(v-W) \qquad (4)$$

$$S = (\frac{v+W}{2})(r-R) \qquad (5)$$

where I is an industry selection; S is a stock selection; *low case* letters relate to a portfolio; *upper* case letters relate to a benchmark; v and W are weights of some sector in the portfolio and benchmark accordingly; r and R are the portfolio's and benchmark's returns accordingly for the same sector.

This model has been derived from the following considerations. The benchmark and portfolio are two entities to be compared by methods of attribution analysis. With this regard they are equal entities in all respects, as the mere fact of their comparison implies. We can compare objects or phenomena of the same class or type. The results of such a comparison should not depend on whether we compare the first entity to the second, or the second entity to the first. These results can have different algebraic signs because of the direction of comparison, but the results should be equal in absolute values. (They can also be conjugative in a general case. Opposite algebraic signs is a particular case of conjugation.) The attribution model defined by equations (4) and (5) satisfies these symmetry requirements. If the portfolio's data and benchmark's data are swapped, then attribution parameters will differ only in algebraic signs.

Another feature of this model is the symmetry of the appropriate formulas that should support the functional symmetry. Formulas (4) and (5) are symmetrical in their form too. On the other hand, attribution models such as the aforementioned BHB model or Brinson-Fachler model (Brinson, 1985) produce asymmetrical results for the swapped data, while their formulas have also asymmetrical appearance. BHB model includes four benchmark's parameters and only two portfolio parameters. Brinson-Fachler model includes five benchmark's parameters and two portfolio's parameters, which confirms their asymmetrical nature.

4. A global attribution model

Our analysis will be based on formulas (2) and (3) for the global investment return. We apply formulas (4) and (5) toward the currency and market attribution analysis *separately*. Before we begin this analysis, we have to clarify one more issue in order to do the global attribution analysis properly. Namely, we should answer the question, how one should define the weights for the global attribution. The answer is that we should use the initial weights of assets within the investment portfolio, otherwise (3) will invalid, because its derivation is based on this assumption, which is the cornerstone of the attribution analysis in general.

A numerical example in table 1 illustrates application of methods discussed in this section. Here, we introduce a portfolio and a benchmark with two segments, which can be market segments (energy, staples) or country segments, or base on other type of classification. We consider a general case when different sectors can be traded in different foreign markets. There is also another dimension of generality in this model, which allows for different currency exchange rates, used for the portfolio and benchmark transactions. In many instances these exchange rates are assumed as equal. However, the business situation when exchange rates are different is possible. Suppose we want to compare two different portfolios. Both portfolios had transactions in different markets for the same sectors, or maybe the time of transactions was different. In this case, most likely the exchange rates are different as well. Such general approach significantly enhances the model's flexibility and allows using it for a wider range of practical business scenarios. The case of equal exchange rates is a particular case of this general scenario.

Using (2), we can define market and currency returns both for the portfolio and benchmarks as follows.

$$R_C = e_f e_b - 1 \tag{6}$$

$$R_M = \frac{E_L - B_L}{B_L} \tag{7}$$

where index "*L*" corresponds to local currency. For simplicity, we assume that the market return is already found using (7). Usually, the benchmark's market return is already known as a percentage value, when the benchmark is some index.

Table 1. Sample data used for a global attribution analysis.

	Portfolio Weight, %	Benchmark Weight, %	Portfolio. Forward currency exchange rate	Portfolio. Backward currency exchange rate
Segment 1	30	45	1.1	1/1.05
Segment 2	70	55	1.5	1/1.6
Total	100	100		

Table 1. continues

	Benchmark. Forward currency exchange rate	Benchmark. Backward currency exchange rate	Portfolio market return, %	Portfolio currency return, %
Segment 1	1.12	1/1.06	10	4.76
Segment 2	1.52	1/1.59	-3	-6.25
Total			R_{pM}	R_{pC}

Table 1. continues

	Benchmark market return, %	Benchmark currency return, %	Portfolio, total return, %	Benchmark, total return, %
Segment 1	6	5.66	15.236	12.0
Segment 2	2	-4.4	-9.0625	-2.488
Total	R_{bM}	R_{bC}	-1.773	4.0316

Table 1 lists values R_{pM} , R_{pC} , R_{bM} , R_{bC} as indefinable. These notations denote that at this point we cannot calculate separately the portfolio's total market return and portfolio's total currency return based on segments' returns, and we cannot do this for the benchmark as well. We will explain shortly from mathematical perspective why this happens. However, we can calculate a total return *for the same segment* using formula (3) that allows combining segment's market and currency returns into a total segment's return. This way we can find the portfolio's market return and portfolio's currency return for each segment *separately*. By the same token, we can find a total return for each segment of the benchmark. These values are presented in the last two columns, in the first two rows. Only then, knowing total returns for each segment, we can find the weighted total returns both for the portfolio and benchmark as follows.

$$R^T = R^T_{seg1} W_{seg1} + R^T_{seg2} W_{seg2} \tag{8}$$

where index "*T*" refers to total return.

This is how the portfolio's total return (-1.773) and the benchmark's total return (4.0316) were calculated. It would be nice to find such total market returns R_{pM}, R_{bM}, and currency returns R_{pC}, R_{bC} listed in table 1, but we cannot do this correctly from a mathematical perspective. The culprit is the effect similar to attribution's interaction. Using (3) and (8) we can write the following expression for the total return using segments' returns.

$$\begin{aligned} R^T &= (R^M_{seg1} + R^C_{seg1} + R^M_{seg1} R^C_{seg1}) W_{seg1} + (R^M_{seg2} + R^C_{seg2} + R^M_{seg2} R^C_{seg2}) W_{seg2} = \\ &= (R^M_{seg1} W_{seg1} + R^M_{seg2} W_{seg2}) + (R^C_{seg1} W_{seg1} + R^C_{seg2} W_{seg2}) + R^M_{seg1} R^C_{seg1} W_{seg1} + \\ &+ R^M_{seg2} R^C_{seg2} W_{seg2} \end{aligned} \tag{9}$$

The first two terms (on the second line of (9)) represent an additive part. The first term $(R^M_{seg1} W_{seg1} + R^M_{seg2} W_{seg2})$ is the addition of weighted market returns for two segments, while the second term $(R^C_{seg1} W_{seg1} + R^C_{seg2} W_{seg2})$ represents the addition of segments' currency returns (either for a portfolio or benchmark). If we had only these two terms, then the simple approach for the weighted average returns would be legitimate. However, there are two other terms $R^M_{seg1} R^C_{seg1} W_{seg1}$ and $R^M_{seg2} R^C_{seg2} W_{seg2}$, in which the market returns and currency returns are multiplied. So, there is no way to separate the total market and total currency returns in (9). Equation (3) has to be satisfied, because the market and currency returns together have to comply with this equation, and (9) is a consequence of equation (3). So, we cannot decompose the total portfolio return into a total market return and a total currency returns mathematically correctly. The same is valid for the total benchmark return.

If one wants to fill out the missing returns in table 1 denoted by variables, then he can do some approximation transformations in order to circumvent the nature of the phenomenon. For example, the sum of the last two terms in (9) can be redistributed between the first two terms (each in its brackets). It should be done proportionally to the value of these bracketed terms representing the weighted market return and currency return accordingly. It will be an artificial construct of course, but it might work for certain applications. In the same way, the present attribution models, such as the BHB model, work for attribution analysis despite the presence of an interaction effect. However, it will be an *approximation* of the real phenomenon. Error valuation should be done in each particular application.

Another approach might be to use a matrix algebra applied to (9), and find a canonical form of this expression, similar to what has been done in (Shestopaloff, 2009) for a symmetric arithmetic model.

So, the overall result of the conducted research is positive. The introduced method allows finding a total market return and a total currency return *separately*. It allows doing a complete attribution analysis on a sector level. We cannot find the values of total returns from the sector's returns. However, this is not important because we can do this in the base currency anyway. So, we would not gain much even if such functionality is available.

The next thing to do is to find the attribution parameters for the data from table 1 using the aforementioned symmetrical attribution model. After substitution of segments' weights and appropriate rates of return from table 1 into formulas (4) and (5), we obtain the following results presented in table 2.

Table 2. Attribution parameters for the data from table 1.

	Industry selection	Stock selection	Country selection	Currency selection
Segment 1	-0.012	0.015	-0.007815	-0.003375
Segment 2	-0.00075	-0.03125	-0.008	-0.01156
Total	**-0.01275**	**-0.01625**	**-0.015815**	**-0.014935**

The manager's overall performance is not impressive according to these results. If one analyzes the input data, he will get a similar feeling without computing the attribution parameters. We do not do comparison with the classic attribution model because of the high value of interaction effect embedded into this model for the analyzed set of data.

5. Currency hedging and multiple currency transactions

The next issue is currency hedging. Hedging is often used as a protection against undesirable movement of exchange rates. It can be done using variety of financial instruments including the ones with a very high leverage. These instruments are extremely risky if no simultaneous reverse position is taken. The positions taken in a hedging are such that in whatever direction the exchange rate moves, the hedger will get the price within the range he secured. In reality things are not so simple, as recent example with Brazilian companies shows, when the supposed to be protective currency hedging worked against the hedgers. Obviously, there is a price for such a convenience. This is the price of hedging contracts plus broker's fees and exchange commissions. When such a safety net arrangement includes only the local and base currency transactions, then this is hedging. If a more sophisticated transaction is used, for instance, when a third currency is included into transaction, then this scenario is called cross-hedging.

Whatever type of currency hedging is used, the outcome always can be reduced to a scenario when the beginning investment value and the ending value are evaluated in the base currency. Hedging expenses can be subtracted from these amounts at the appropriate transaction phases, so that technically this is not a complicated procedure. We will concentrate on calculating currency return on hedging first, and also consider a special case of active currency management, when a fund manager is seeking active currency returns. Of course, these results are applicable to the case when no explicit hedging is involved, but when someone invests into foreign markets just exchanging the currency on the spot.

Let us analyze the separation of currency and market returns. The market return is defined as usual, in the local currency. The currency return can be found as follows. One approach is to assume that we do only the currency exchange without buying any local stock, which is also, in fact, the case of active currency management. We will operate in terms of exchange rates only, in order to find the total currency return. As we mentioned earlier, the result should not depend on the transaction type; that is either the transaction is done with hedging or without it (except for some adjustments because of the hedging fees, which we consider later). For example, let us consider the chain of currency exchange operations presented in table 3.

Table 3. Sequence of currency exchange transactions

Transaction number	Currency	Exchange rate	Amount
Base currency	CAD		**1.0**
1	CAD - USD	1.016	1.016
2	USD - EUR	0.68	0.691
3	EUR - GBP	0.72	0.497
4	GBP - CAD	2.07	1.03
Base currency	CAD		**1.03**

According to (2) and (6), the total currency return is defined as follows.

$$R_T = \left[\prod_{j=1}^{j=N} e_j \right] - 1 \qquad (10)$$

where e_j is the exchange rate for the *j*-th transaction.

If someone is exchanging the beginning amount of money *B* into some currency, not necessarily into the original one, then the ending amount *E* is defined as follows.

$$E_N = B \times \left[\prod_{j=1}^{j=N} e_j \right] \qquad (11)$$

Formulas (10) and (11) describe, in fact, also the case of active currency management without hedging fees. Now, we will take into account the hedging fees. The hedging can be done through three-month currency forwards, but it can also use the spot market, options or futures. Our idea is to avoid absolute values, such as transaction and hedging costs valued in some currency, and to remain within the realm of exchange rates using some relative values. We can do this by counting the cost of hedging and transactions as a relative value influencing the resulting exchange rate. For instance, we always know the price of option contract, transaction costs associated with each contract, and we know also the contract amount. So, in both cases, either we exercise or do not exercise the options, the costs are known and can be related to the involved total currency amount as a relative value, as a multiplier. Suppose such relative transaction and hedging costs are equal to t_j (sum of fees and hedging costs divided by the total currency amount). Then, we can take into account these costs by transforming formula (10) as follows.

$$R_T = \left[\prod_{j=1}^{j=N} e_j \prod_{k=1}^{k=K} (1 - t_k) \right] - 1 \qquad (12)$$

Accordingly, (11) transforms as follows.

$$E_N = B \times \left[\prod_{j=1}^{j=N} e_j \prod_{k=1}^{k=K} (1 - t_k) \right] \qquad (13)$$

This approach, when all parameters are relative, is beneficial from the perspective of analytical studies, which acquire in this case a great deal of flexibility in designing and analyzing different investment scenarios.

6. Conclusion

This article introduced a model of global attribution. It considered in detail the problem of finding currency and market rates of return both for the portfolio and benchmark. Mathematical formulas have been derived and analyzed from the perspective of applicability in practical situations. A numerical example has been used to demonstrate the method's application to a particular investment scenario. The model allows finding the total sector's returns when the market and currency returns have to be combined. It also allows combining sectors' returns into a market return, or into a total currency return. It was proved that it is impossible to correctly find the overall investment return through the total sector returns, or through the total market return and a total currency return. However, this is not a critical issue for the global attribution analysis because these values can be found directly in the base currency.

A symmetrical arithmetic attribution model has been used as a base for the introduced global attribution model. As a result, the newly global attribution model provides better objectivity because it does not have an interaction effect, or other noise factors.

Currency hedging and multiple currency transactions have been considered and incorporated into the proposed global attribution model. The proposed approach allows taking into account both the influence of currency exchange rates and transactional and hedging fees to the total return. This approach and derived formulas embrace also the case of active currency management.

Overall, the introduced model represents a comprehensive and practical mathematical vehicle that can be used in diverse practical applications and advanced analytical studies.

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