GEOMETRIC ATTRIBUTION MODEL AND A SYMMETRY PRINCIPLE

Yuri Shestopaloff,

Introduction

This article considers an existing geometric attribution model and introduces a new one. Analysis is based on a conceptual framework grounded on a symmetry principle. This principle closely relates to a more general principle of invariance to permutations, well developed in mathematics. The framework consists of a few fundamental principles, or concepts, validation tools and verification criteria. Geometric attribution methods are analyzed within this framework. Some inadequacies of the present attribution method have been discovered, and a new model has been developed and verified by the introduced validation criteria. Numerical examples are presented, and they confirm the analytical results.

Attribution analysis is a well established area in the investment industry. Foundations were developed a while ago and much of the present developments owe to approaches introduced in Brinson (1985), Brinson (1986). These works set the ideology of attribution analysis. The main idea of the authors' approach is to consider benchmark and portfolio returns as some opposite entities, and introduce intermediate value that serves as a threshold in between the boundary values. This threshold value is used as a reference point to decompose the contribution to relative return into a few attribution parameters. For example, in case of equity portfolios these are industry selection and stock selection.

The specific aspects of these approaches were well analyzed in many publications, improvements have been introduced, and substantial new developments followed. A geometric attribution has been added and became a prominent alternative to arithmetic attribution models. The details can be found in Burnie (1998), Menchero (2001), McLaren (2001), Bacon (2002). Linking attribution parameters over time periods is another important application area of attribution methods, see Frongello (2002), Menchero (2004). A book of Spaulding (2003) covers the subject comprehensively.

There are at least two issues with existing approaches. Firstly, the threshold value has to be chosen objectively based on well defined and robust criteria. Secondly, validation tools should be available to allow verification of newly developed attribution models. Present attribution methods lack the criteria for choosing threshold and do not provide validation tools. Heuristic component is prominently present in attribution models. For instance, the choice of a threshold value is done mostly intuitively, without rigorous criteria.

The pioneering works Brinson (1985), Brinson (1986) still deserve the highest respect and credit. It is always difficult to be a pathfinder. However, the new requirements and new information appear all the time. So, time comes to review some principles of the attribution analysis in order to provide more objectivity to this discipline, and equip it with the better grounded methodological approaches.

1. Contribution

We will define contribution first in order to have a clear understanding of the foundation the attribution analysis is based upon. This notion literally defines the contribution of a chosen group of financial instruments (such as a collection of stocks or an individual instrument) into the total portfolio return or benchmark return. Contribution formulas are well known. However, it is better to do derivation from scratch to understand the subject better because most works omit this issue. Suppose that a portfolio consists of *N* groups of financial instruments (it can be separate securities, industry sectors, etc.). Then portfolio return is defined as follows.

$$R_p = \frac{E_p - B_p}{B_p} = \frac{\sum_{i=1}^{i=N} B_i(1+R_i) - \sum_{i=1}^{i=N} B_i}{B_p} = \frac{\sum_{i=1}^{i=N} \left[B_i(1+R_i) - B_i\right]}{B_p} = \\ = \sum_{i=1}^{i=N} \frac{B_i R_i}{B_p} = \sum_{i=1}^{i=N} W_i R_i \qquad (1)$$

where R_p is portfolio return; R_i is a group's return; W_i is the group's weight within the whole portfolio; *i* - index denoting the group number; index *p* relates to portfolio; *B* denotes the beginning market value; *E* is the ending market value.

Expression (1) resolves the issue of what weight has to be used in performance attribution. Expression (1) proves that it has to be weight related to beginning market value.

2. Arithmetic attribution

Arithmetic attribution is based on contribution defined via the beginning market values. We will use the following notations. Capital letters will refer to the benchmark, while lower-case letters will refer to the portfolio. Rate of return will be accordingly *R* and *r*, weights *W* and *v*, differences between rates and weights are defined as follows: $d_v = v - W$, $d_r = r - R$. Then the difference between contribution into portfolio's return by some group of stocks and contribution of the same group into the benchmark return is defined as follows.

$$D = rv - RW = (R + d_r)(W + d_v) - RW = Rd_v + Wd_r + d_v d_r = \\ R(v - W) + W(r - R) + (r - R)(v - W) \qquad (2)$$

where notations for simplicity relate to one group of stocks, so there are no indexes in (2).

Parameter D is called contribution to relative return (CRR). As we can see, this is the difference between weighted returns of the same group of securities in the portfolio and benchmark. Weights correspond to the relative beginning market value of this group within the portfolio and benchmark. A geometric attribution, on the contrary, is based upon definition of contribution to relative return that exploits the ending market values, while the returns themselves are still defined by the same contribution formula (1).

Although CRR can represent the difference between total portfolio and benchmark returns, it makes sense to use another term in this case. It can be the often used optimistic term “excess return”, “total returns’ difference” or just “returns’ difference” when the context is clear.

The first term in expression (2) is well known as *industry selection or timing*, the second term as *stock selection*. These parameters are defined in Brinson (1986) model. It is often known as BHB model, after the authors’ last names Brinson, Hood and Beebower. We also have a third term which is called *interaction*. Its business interpretation is murky, given the variety of opinions we encountered.

The interaction is a product of differences in weights and returns between the portfolio and benchmark. This term can be interpreted as a noise because of its uncertain business meaning. Ideally, as long as we factor the difference in returns between the benchmark and portfolio into stock selection and industry selection, we would like these components to constitute the whole difference. The presence of interaction jeopardizes the factoring idea.

3. Specifics of attribution models

A typical feature of present attribution models is its inclination to use benchmark parameters as the primary values. Portfolio and benchmark are entities of the same type. Suppose we present to somebody two data sets, one belonging to a benchmark and the other one to a portfolio. If we do not identify what entity each data set represents, there is no way to identify them. On the other hand, if we would like to do attribution comparison for two portfolios, which one should be given the benchmark role? Or, if we would like to do attribution analysis for two benchmarks, then which one is the primary? We have to make such decision with present attribution methods, otherwise we cannot use them for these scenarios. However, such situations are both practical and valid from the attribution analysis standpoint. The concept of attribution analysis does not impose any limitations with this regard.

So, there are no justifiable reasons to tip the balance into a benchmark’s favor. When we compare entities of the same type, they should be treated equally to preserve the balance and provide an objective comparison. The purpose of comparison is to find the *difference* between them. Comparison methods should have a property of symmetry or invariance to permutations. The result should not depend whether we compare the first entity to the second or the second entity to the first when we are interested only in the *difference*. The results of comparison may be conjugated (for instance, have different algebraic signs), but they have to relate to each other if this is a valid comparison method. There are objects and methods where this invariance to permutations is not preserved. However, attribution objects such as portfolio and benchmark are characterized by exactly the same data sets and do not belong to this category.

If we analyze the present attribution methods from the perspective of invariance to permutations and symmetry, then we find the following. Stock selection and industry selection in BHB model are defined by six parameters in formula (2). Four parameters belong to benchmark, while only two relate to portfolio. Brinson-Fachler model has five benchmark’s parameters, and only two belong to portfolio. Geometric attribution parameters are asymmetrical as well.

Here is another consideration. We can compare objects of the same type, or the same properties of different objects. We can compare a certain frog and a wolf in weight because both have the same property - weight. We cannot compare their natural hibernation periods because wolves do not hibernate. The present attribution methods do not treat compared properties equally. In fact, they compare slightly different properties despite the original intention. The analogy can be comparing the volume of the first object to the mass of the second. If we switch the objects, then, if we apply the present attribution ideology, we have to compare the volume of the second object to the mass of the first object. Obviously, the results will be different in absolute values. In the case of attribution models the situation is not as obvious though at its core it is very similar. The results of models' application to the original benchmark and portfolio data sets, and to the swapped data sets are more revealing with this regard. These results are asymmetrical. This fact implies that present models do not exactly evaluate the same properties when they are applied to the attribution comparison of benchmark and portfolio.

4. Introducing principles of symmetry and data swapping

Present attribution models, such as BHB, Brinson-Fachler and geometric model refer to the benchmark as a reference point, or the reference entity. It might be considered as a logical thing to do at first glance, because the benchmark functions as a referral entity by definition. This fact has a psychological effect resulting in the primacy of the benchmark, as we have seen in our analysis. What really matters is *the difference* between some derivatives of these two entities. However, this implies that derivatives themselves have to be defined properly, in order to be able to objectively represent the difference between these entities. The objective difference has to fit into a symmetrical context, where direction of comparison should not influence the absolute value of this difference. We should be able to swap the entities within the right symmetrical context, and obtain the same results, maybe with the opposite signs or conjugative in general.

We have just formulated some important general principles that apparently have a wider area of applicability than attribution analysis. They can be refined as follows in order to emphasize their generic nature.

General comparison principles related to finding difference between objects of the same type or identical properties of objects.

Req. No.	Requirement
1	The correct comparison method has a property of symmetry, which assumes that the absolute value of the result of comparison should not depend on the direction of comparison, whether the first entity is compared to the second, or vice versa.
2	A valid comparison method should allow swapping of compared entities, and the results of comparison for such swapped and non-swapped entities should be conjugative.

3	The correct symmetrical comparison method excludes the presence of noise factors and leaves only the meaningful parameters.
4	Functional symmetry of a comparison method assumes a symmetrical form of its mathematical implementation.
5	Initial symmetrical data sets produce symmetrical resulting parameters.

The violation of these comparison rules with regard to the same type entities or identical properties leads to asymmetrical and thus incorrect results. Interaction and other noise factors in the present attribution models are the consequence of such *asymmetrical* comparison parameters for *symmetrical* entities.

5. A symmetrical data model

We will explore the above considerations using a numerical example. Table 1 contains a purposely chosen symmetrical portfolio and benchmark data. Developing attribution models today is a fairly subjective process. Introduction of such a *symmetrical data model* creates a very good reference point when developing attribution methods. Data interpretation is straightforward in this case because the right model should preserve the symmetry of obtained values. Such symmetrical data set can be an enhancement of what we introduced in Table 1 and accommodate specific requirements. Given the symmetry of input data, it is common sense that objective attribution parameters should be symmetrical as well.

Table. 1. Sample symmetrical portfolio

Group No.	R	r	W	v
	0.3	0.2	0.2	0.3
	0.3	0.2	0.3	0.2
	0.2	0.3	0.3	0.2
	0.2	0.3	0.2	0.3
Total	**0.25**	**0.25**	**1.0**	**1.0**

In the next section we will begin analysis of a geometric attribution model. Table 2 presents attribution parameters produced by this method for a symmetrical data set. The results are asymmetrical, which implies that the model did not pass the symmetry test. The next test would be swapping of the benchmark data and portfolio data, but only if the symmetrical data set produced symmetrical attribution parameters. However, in this case the first test has discovered the method’s asymmetry.

Table 2. Attribution parameters obtained for the symmetrical data set by a geometric attribution method.

Group No.	Industry Sel	Stock Sel.	Interaction
1	0.004	-0.024	0

	2	-0.004	-0.016	0
	3	0.004	0.016	0
	4	-0.004	0.024	0
Total		0.0	0.0	0.0

6. Geometric attribution

In this section we consider a geometric attribution model in detail. We will develop a new geometric attribution model using the concepts and rules introduced in the previous sections, in order to overcome the limitations of existing approaches. The reader can acquire a familiarity with existing approaches in Spaulding (2003), Burnie (1998), Menchero (2001), McLaren (2001), Bacon (2002).

In geometric attribution the contribution to relative return D (the weighted difference in returns on a group level), and the difference between the portfolio return and benchmark's return (excess return) are defined as follows.

$$D = \frac{1+r}{1+R} - 1 \qquad (3)$$

where as before values r and R relate to rate of return for a portfolio and benchmark accordingly.

Thus introduced relationship is not a very obvious one. We will rewrite it as follows in order to clarify the business meaning.

$$D = \frac{r-R}{1+R} = \frac{\frac{e-B}{B} - \frac{E-B}{B}}{1 + \frac{E-B}{B}} = \frac{e-B-E+B}{B+E-B} = \frac{e-E}{E} = \frac{e}{E} - 1 \qquad (4)$$

where e and E are the portfolio's and benchmark's ending values; B is the beginning market value that is chosen to be the same for the portfolio and benchmark (we can do this based on a definition of rate of return).

So, geometric contribution to relative return can be considered as the ratio between the ending market values of groups of securities included into the portfolio and benchmark minus one. The same statement is true for the excess returns. This notion itself is clear and meaningful. Essentially we compare the ending market values of securities groups' in the portfolio and benchmark when the beginning market values are the same.

Formula (3) can be used to find the geometric CRR. The returns themselves are found using the same definition of contribution as we used for arithmetic attribution.

Our goal is to decompose the contribution to relative return defined by equation (3) into two or more values meaningful from the attribution perspective. We have a choice of sums and products. Arithmetic attribution uses sums. Geometric attribution uses products, in the following form.

$$D+1=\frac{1+R_P}{1+R_B}=\frac{1+R_P}{1+R_S}\times\frac{1+R_S}{1+R_B} \tag{5}$$

Here and below we will use the following notations. R_P is the portfolio's total return; R_B is the benchmark's total return; index *j* relates to the *j*-th component, that is a group of securities; v_j is the weight of this component in the portfolio; W_j is the weight of this component in the benchmark.

Value R_S is introduced as follows. (It is known as seminotional return.)

$$R_S=\sum_{j=1}^{N} v_j R_j \tag{6}$$

Upper case letters relate to the benchmark while *lower case* letters relate to the portfolio.

This threshold value has been introduced without strong grounds, just based on the precedent of the BHB attribution model. We know the weakness of this reference point, which is its asymmetry. The same asymmetry has been carried to the geometric attribution model without critical analysis.

The total industry selection and stock selection are defined accordingly by the following formulas.

$$I=\frac{1+R_S}{1+R_B}-1 \tag{7}$$

$$S=\frac{1+R_P}{1+R_S}-1 \tag{8}$$

We will rewrite equation (6) in order to emphasize that geometric linking does not include noise terms like interaction. That is

$$D+1=(I+1)\times(S+1) \tag{9}$$

This is a nice feature from a practical point of view. It is a direct consequence of the definition of geometric attribution. The returns' difference decomposes into two independent factors, each representing the required business functionality. So, in geometric attribution, defined by equation (4), we do not have noise factors.

We have to note that definition of geometric attribution by formulas (4), (6) is just one of many. Similarly, one could define R_S as it is done in formula (10) below or in some other way. However, none of these definitions include a clear explanation of why this particular definition was chosen. All of them are based mostly on heuristic considerations.

$$R_S = \sum_{j=1}^{N} W_j r_j \qquad (10)$$

So, the choice of this threshold value in reality is subjective. It is not unlike setting a divider between two boundaries, which is essentially what R_S is. This is a worrisome fact that must be taken seriously because geometric attribution is gaining ground. This is why we decided to reiterate this fact one more time. *The divider presently used in the geometric attribution model is not a well grounded value, but rather is arbitrarily and heuristically chosen.* On the other hand, this value has crucial implications to the results obtained using the geometric attribution model.

The next logical step would be to define attribution parameters for an individual group. Industry selection is computed using the following formula.

$$I_j = (v_j - W_j) \times \left[\frac{1 + R_j}{1 + R_B} - 1 \right] \qquad (11)$$

If we treat the last factor as a returns' difference (which is true, this is just *geometric* returns' difference), then this equation is conceptually very similar to Brinson-Fachler attribution model, Brinson (1985). If fact, equation (11) is designed to evaluate the industry selection value relative to the benchmark's total return. It is not surprising that industry selection in geometric attribution model is very close to the value of Brinson-Fachler's industry selection. The smaller R_B is, the lesser is the difference.

Stock selection (also timing, asset allocation) is defined by the following formula, which is not exactly intuitive.

$$S_j = v_j \times \left[\frac{1 + r_j}{1 + R_j} - 1 \right] \times \left[\frac{1 + R_j}{1 + R_S} \right] \qquad (12)$$

Other models with different stock selection definitions are available. In all these approaches, the major drawback is the arbitrariness of the threshold value R_S.

7. Developing a new symmetrical geometric attribution method

We will present a new geometric attribution method in this section, calling it the symmetrical geometric attribution (SGA) model. It overcomes some drawbacks of the present geometric attribution models. On the other hand, it provides an example, how general considerations based on the symmetry principle, and introduced in this work, can be implemented. These considerations implement the rules for comparison algorithms.

1. The use of a symmetrical data set and swapping of benchmark data and portfolio data as validation tools.
2. Preserving the formal symmetry of quantitative implementation.
3. Requiring that results obtained by swapping the benchmark's and portfolio's data are conjugate.

4. An objective choice of a threshold value for factoring the contribution to relative return into meaningful attribution parameters.

We should point out the contradiction between the multiplicative relations between attribution characteristics on the total level and the additive nature of attribution parameters on the group level. This contradiction cannot be overcome, so we should make a choice of what feature we will preserve. The multiplicative relations between attribution parameters on the total level are more valuable to geometric attribution, so we sacrifice the additive feature which would allow obtaining the total attribution values from the groups' attribution parameters by simple summation.

The first issue is that we have to guarantee the relation

$$D + 1 = (I + 1) \times (S + 1) \tag{13}$$

In geometric attribution the contribution to relative return is defined by expression (13).

So, we should find some approach to resolve this issue. We will define the stock selection for a group level as follows.

$$S_j = \left(\frac{v_j + W_j}{a}\right) \times (r_j - R_j) \tag{14}$$

where a is some constant to be found.

Industry selection will be defined as follows.

$$I_j = \left(\frac{r_j + R_j}{a}\right) \times (v_j - W_j) \tag{15}$$

Formulas (14) and (15) defining the attribution parameters satisfy the mathematics symmetry and data swapping requirements. Let us find the attribution characteristics at the total level, industry selection and stock selection, using the introduced attribution parameters. This can be done as follows.

$$\begin{aligned} S &= \sum_{j=1}^{j=N} S_j = \frac{1}{a} \sum_{j=1}^{j=N} (v_j r_j + W_j r_j - v_j R_j - W_j R_j) = \\ &\frac{1}{a}\left(\sum_{j=1}^{j=N} v_j r_j - \sum_{j=1}^{j=N} W_j R_j - \sum_{j=1}^{j=N} (v_j R_j - W_j r_j)\right) = \frac{1}{a}(P - B - D_S) \end{aligned} \tag{16}$$

where $D_S = \sum_{j=1}^{j=N} (v_j R_j - W_j r_j)$, P and B are total returns of portfolio and benchmark accordingly.

A similar transformation of equation (15) produces the following result for the integral industry selection.

$$I = \frac{1}{a}(P - B + D_S) \tag{17}$$

Let us substitute the right sides of (16) and (17) into equation (13). This looks as follows.

$$\left(\frac{1}{a}(P - B + D_S) + 1\right) \times \left(\frac{1}{a}(P - B - D_S) + 1\right) = \frac{1+P}{1+B} \tag{18}$$

Doing the appropriate transformations of equation (18) we obtain a quadratic equation. Solving it with respect to *a* results in the following two solutions

$$a_{1,2} = \frac{(P-B) \pm \left(\frac{(P-B)^2(1+P) - D_S^2(P-B)}{1+B}\right)^{\frac{1}{2}}}{\frac{P-B}{1+B}} \tag{19}$$

This solutions have some restrictions which we have to explore. First of all, the discriminant in formula (19) cannot be negative. Secondly, the discriminant cannot be equal to zero. This happens when $B = P$. The last restriction is $B \neq -1$. Let us check the discriminant first.

$$(P-B)^2(1+P) - D_S^2(P-B) \geq 0 \tag{20}$$

This inequality includes two cases. The first one is when $P \leq B$. In this case inequality (20) transforms into the following.

$$(P-B)(1+P) \leq D_S^2 \tag{21}$$

This inequality is true because $P \geq -1$, consequently $(1+P) \geq 0$, and $(P-B) \leq 0$, because we assumed $P \leq B$. Thus the left part is always non-positive, while the right side is always non-negative.

The second case when $P \geq B$ is not obvious. Transformation of equation (20) in this case produces the following inequality.

$$(P-B)(1+P) \geq D_S^2 \tag{22}$$

Though the left side of equation is non-negative, it is not clear if it exceeds the right side, which is always positive. Let us consider the same group of financial instruments within the portfolio and benchmark. This group has the same weight in the portfolio and benchmark. In this case we can rewrite inequality (22) for this group only as follows.

$$vr - WR + v^2r^2 + vrWR - v^2R^2 - W^2r^2 \geq 0 \tag{23}$$

Assuming $v = W$ we can transform expression (23) into the following inequality.

$$(r - R)(1 + vR) \geq 0 \qquad (24)$$

The second term in (24) is always non-negative because a group's weight cannot exceed one unit, and rate of return cannot be less than -1. The first term is non-negative only when $r \geq R$. This restriction actually complies with our assumption that $P \geq B$ if the group constitutes the whole portfolio. Otherwise we have to make such an assumption with respect to each group. If this is the case, then inequality (22) is fulfilled for the whole portfolio.

In total we discovered that discriminant (20) is non-negative for all $P \leq B$. We also found a sub-domain for the case when $P \geq B$, though we did not cover the whole domain. So, the direct algebraic approach did not resolve the issue completely; and we should take into account more factors associated with the problem. One of these factors is the method's symmetry. This results in a symmetry of mathematical implementation (19) with regard to benchmark and portfolio parameters. So, if $P \geq B$, we just have to swap the benchmark and portfolio data treating benchmark as portfolio and vice versa. Repeating computations (16), (17) and (18) for this scenario, we obtain the formula mirroring (19), in which *B* is substituted by *P* and *P* by *B*. The nominator of a discriminant for the appropriate quadratic equation will be

$$(B - P)^2(1 + B) - D_S^2(B - P) \geq 0$$

We proved already that it is a non-negative value for all $B \leq P$. This completes the proof. So, discriminant is non-negative for *all* combinations of returns.

We will formulate the "*Discriminant Theorem*" as follows.

> *The inequality* $(P - B)^2(1 + P) - D_S^2(P - B) \geq 0$ *is always fulfilled for any portfolio and benchmark input data, where P and B represent the portfolio's and benchmark's total returns, and* D_S *is defined as follows:* $D_S = \sum_{j=1}^{j=N}(v_j R_j - W_j r_j)$.

We also mentioned the restriction $B \neq -1$. This condition is not a consequence of the method itself but rather this restriction is a consequence of the geometric attribution approach in general.

The next restriction to be considered is $B = P$. In this case equation (19) produces $D_S = 0$ and we cannot find a precise value of *a*. The solution does not exist. The correct approach is to find the limit of (19) when $(P - B)$ approaches zero. This limit can be found as follows.

$$a_{1,2} = \lim_{P-B\to 0} \frac{(P-B) \pm \left(\frac{(P-B)^2(1+P) - D_S^2(P-B)}{1+B} \right)^{\frac{1}{2}}}{\frac{P-B}{1+B}} =$$

$$\lim_{P-B\to 0}(1+B)\left(1 \pm \sqrt{\frac{1+P}{1+B} - \frac{D_S^2}{(P-B)(1+B)}}\right) = (1+B)\left(1 \pm \sqrt{\frac{1+P}{1+B}}\right) =$$

$$= (1+B) \pm \sqrt{(1+P)(1+B)}$$

(25)

The nice thing about result (25) is its continuity. It means that if we change the input data by a small amount, the value of the coefficient will also change by a small amount, *of the same order* of magnitude. So, we do not have any jumps in expression (19) which we consider to be a function of group weights, groups' returns and total returns.

The next note is devoted to the two roots of expression (19). Given the nature of definitions of attribution parameters, we should choose the largest positive root. Otherwise, the relation of attribution parameters will be preserved, but their absolute value will be larger. Essentially, this is a matter of convenience for consistency in interpreting the results. So, we should rewrite the solution (19) as follows. First, $B \neq -1$. Then we have

$$a = \frac{(P-B) + \left(\frac{(P-B)^2(1+P) - D_S^2(P-B)}{1+B} \right)^{\frac{1}{2}}}{\frac{P-B}{1+B}} \qquad \text{when } P \neq B$$

$$a = (1+B) + \sqrt{(1+P)(1+B)} \qquad \text{when } P = B$$

(26)

Another proof of Discriminant Theorem

Here is another approach to prove the "Discriminant Theorem" based on more general considerations. The transformations we performed to obtain the quadratic equation (18) mathematically are not equivalent. It means that we can *add* more solutions when doing these transformations, but we *cannot lose* the correct solution. It will be mixed with additional roots of the quadratic equation, but it will be *definitely* among the available solutions.

The right side of equation (18) is non-negative, and it is *always* defined. There are no such values of its parameters that make it invalid (except *B* = -1, which is already excluded by the definition of geometric contribution to relative return). Right sides of equations (16) and (17) are also *always* valid expressions in the domain of their parameters. So, equation (18) is *always* valid in the domain of its parameters. Then it necessarily follows from the validity of this equation that a *valid* value of the parameter *a* must *always* exist, which is impossible if we assume that the discriminant can be negative for some combination of parameters. We encounter the contradiction that effectively completes the proof of “Discriminant Theorem”.

8. Comparing the new symmetric geometric attribution model with the present geometric attribution method

We have seen already that the present geometric attribution model produces asymmetrical attribution parameters from the symmetrical data set (Table 2). Table 3 presents the same parameters for the swapped data sets, meaning that swapping portfolio and benchmark data (or calling the benchmark the portfolio and the portfolio the benchmark). Table shows that the symmetry of attribution parameters is also not preserved in this case.

Table 3. Attribution parameters obtained using geometric attribution for the swapped portfolio and benchmark data

Group No.	Industry Sel	Stock Sel.	Interaction
1	0.004	0.016	0
2	-0.004	0.024	0
3	0.004	-0.024	0
4	-0.004	-0.016	0
Total	0.0	0.0	0.0

Application of the new geometric attribution model to symmetrical data set produces symmetrical results, presented in Table 4. All attribution parameters have the same absolute value and the appropriate symmetric algebraic signs. When we compared the results with similar calculations based on a symmetric *arithmetic* attribution model (not discussed in this paper), then the results matched the signs and symmetry, but they were slightly different in absolute values. This is due to the geometric definition of contribution to relative return that employs the ending market values, while arithmetic attribution considers the beginning market values. The absolute contributions to relative return are the same, but in geometric attribution this value relates to the ending market value, and not to the beginning market value.

Table 4. Attribution parameters obtained by the new symmetric geometric attribution model for the symmetrical data set.

Group No.	Industry Sel	Stock Sel.	Interaction
1	0.02	-0.02	0
2	-0.02	-0.02	0
3	-0.02	0.02	0
4	0.02	0.02	0
Total	0.0	0.0	0.0

If we swap the portfolio data and benchmark data in the symmetrical data set, then the results will be exactly the same in absolute value, but algebraic signs will be opposite. So, the model possesses the data symmetry and data swapping features. This result is expected because of the mathematical symmetry of the new method. The numerical example is just a verification of this property using a symmetrical data set. Table 5 presents the data.

Table 5. Attribution parameters obtained by a symmetric geometric attribution model for the swapped portfolio and benchmark using a symmetrical data set.

Group No.	Industry Sel	Stock Sel.	Interaction
1	-0.02	0.02	0
2	0.02	0.02	0
3	0.02	-0.02	0
4	-0.02	-0.02	0
Total	0.0	0.0	0.0

Application of symmetrical geometrical attribution model to arbitrary data sets also confirmed its symmetry with regard to the swapping of benchmark and portfolio data sets. Attribution parameters are the same in absolute value, but their algebraic signs are opposite.

Conclusion

Summarizing the result of this section we would like to emphasize the following. First of all, we introduced the new geometric attribution model based on the geometric definition of contribution to relative return. We demonstrated by example that it provides more objective attribution parameters than the available geometric attribution model. The source of the better objectivity of the new model is a consistent *conceptual* approach we used from the very beginning to the very end. We did not leave a single blind spot that can affect the method's efficiency or restrict its applicability.

The whole conceptual approach we followed in this section can be considered to be a model for the development of other attribution methods. At the beginning we listed the requirements a future attribution model should satisfy. This list can include other requirements and criteria, depending on the particular problem at hand, but the idea remains the same – to follow the conceptual framework proposed in this article for the development of new attribution methods. It just happened that we presented the new geometric attribution model here. It could have been another new attribution method.

Using this new conceptual approach, we also developed the new arithmetic attribution models free from the drawbacks of its predecessors, such as BHB and Brinson – Fachler attribution models. One is called a symmetric arithmetic attribution model (SAA). Its attribution parameters in the previous notations are defined as follows.

$$I = \frac{(r+R)}{2}(v-W) \quad (27)$$

$$S = (\frac{v+W}{2})(r-R) \tag{28}$$

The other is a referential arithmetic attribution model (RAA). This model exploits a reference value *T* similar to the total benchmark return, which is present in Brinson – Fachler model. We use the following substitutions for the new model: $r_T = r - T$, $R_T = R - T$. It means that we consider rates of return adjusted for the reference value *T*. In particular, it can be calculated as arithmetic average of portfolio and benchmark's overall returns. The referential symmetrical attribution model is defined as follows.

$$I_s^T = \frac{(r_T + R_T)}{2}(v - W) \tag{29}$$

$$S_s^T = (\frac{v+W}{2})(r_T - R_T) = (\frac{v+W}{2})(r-R) \tag{30}$$

where I_s^T and S_s^T are accordingly the industry selection and stock selection for the adjusted rates of return.

Reference value *T* is defined by the following formula.

$$T = C \times \frac{B+P}{2} \tag{31}$$

Here *B* and *P* are total returns of the benchmark and portfolio accordingly; *C* is a real number.

We also developed global attribution methods based on symmetry principles. The arithmetic models we developed do not have interaction and other noise effects in principle. These models satisfy all symmetry requirements introduced in this paper. We also developed a notion of the "ideal attribution model". The newly developed arithmetic attribution model, SAA, is the only one we know of that satisfies *all* the requirements of the "ideal attribution model". None of the existing attribution models satisfy this "ideal".

In part, the introduction of the aforementioned new models and the new geometric attribution model discussed in this paper exhibits a heuristic component. We considered this approach as a drawback of the previous models. So, what is the difference? It is substantial. We introduced powerful verification principles and tools that allow separation of correct attribution models from incorrect ones. Have we done something inappropriate, the verification mechanisms would have discovered this error right away. However, we also used a purely analytic approach, without any heuristic components or unjustified assumptions. This is the case with the symmetric arithmetic attribution model. This model has been *derived analytically*, using two independent approaches. One derivation was purely algebraic; the other used matrix algebra. The results proved to be the same, and in both cases these were mathematically rigorous derivations, without any heuristic assumptions. We think that other attribution methods can be derived analytically as well; we just did not attempt to do this.

We think the creation of this family of new attribution models prepares the ground for an eventual standardization of attribution methods. The reason for this is their better objectivity, and an availability of robust conceptual foundations and validation tools. These new concepts can turn attribution analysis from an *art* into a *science*.

REFERENCES

Spaulding, David, “Investment Performance Attribution”, 2003, McGraw-Hill, 254 p.

Brinson G.P., Hood R.L., Beebower G.L. “Discriminants of Portfolio Performance”, Financial Analysis Journal, July-August 1986, p. 39-44.

Menchero, Jose G.,; “A Fully Geometric Approach to Performance Attribution” *Journal of Performance Measurement*, Winter 2000/2001, pp.22-30.

McLaren, Andrew; “A Geometric Methodology for Performance Attribution, *Journal of Performance Measurement*, Summer 2001, pp.45-57.

Bacon, Carl, “Excess Returns – Arithmetic or Geometric?”, *Journal of Performance Measurement*, Spring 2002, pp.23-31.

Frongello, Andrew Scott Bay, “Linking Single Period Attribution Results?”, *Journal of Performance Measurement*, Spring 2002, pp.10-22.

Burnie, J. Stephen, James A. Knowles, and Toomas J. Teder, “Arithmetic and Geometric Attribution”, *Journal of Performance Measurement*, Fall 1998, pp.59-68.

Brinson, Gary P. and Nimrod Fachler, “Measuring Non-U.S. Equity Portfolio Performance”, *Journal of Portfolio Management*, Spring 1985, pp.73-76.

Menchero, Jose G., “Multiperiod Arithmetic Attribution”, Financial Analysts Journal, vol. 2004, Vol. 60, No. 4, pp. 76-91.