CONCEPTUAL FRAMEWORK FOR DEVELOPING AND VERIFICATION OF ATTRIBUTION MODELS. ARITHMETIC ATTRIBUTION MODELS

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Abstract

Article considers the area of investment attribution analysis. It introduces mathematically grounded, objective principles and verification criteria for developing attribution models, which together create a conceptual attribution framework. Existing models have been scrutinized using these new concepts; the known inadequacies of these models with regard to objective representation of attribution parameters, such the interaction effect, have been confirmed. New attribution models have been developed based on introduced principles, thoroughly researched, verified, and compared with existing methods in terms of objectivity. The results proved that the new attribution models provide more objective and meaningful results than existing methods.

Introduction

Attribution analysis, sometimes called as performance attribution, is a developed area of financial mathematics widely used in practical applications. Comprehensive analysis of attribution methods for equity portfolios, and some results referred to in this article, can be found in (Shestopaloff, 2008, 2009a, 2009b). In this article, we concentrate on *general* principles how to develop attribution models, introducing a set of principles and validation criteria. Based on these principles, together composing a conceptual development framework, we introduce and validate two new arithmetic attribution models. Besides, we consider two classic attribution models, first presented in (Brinson, 1985), called Brinson-

Fachler model, and another introduced in (Brinson, 1986), often referred to as the BHB model. Practical application of attribution models is considered in (Bacon, 2008).

Most of the introduced principles and validation criteria are applicable to developing of one method. However, we also introduce one specific criterion called ratio validation criterion that validates one attribution method relative to the other. With this regard, we analyze the geometric attribution model, presented in (Shestopaloff, 2008, 2009), and apply the aforementioned validation criterion to compare this model and the newly developed arithmetic attribution model.

In this paper, we consider mathematical fundamentals rather than particular business practices. The presented attribution framework is open to adjustment to particular business requirements and processes in many different ways. In this regard, we would like to refer the reader to article (Hood, 2005) and following responses, which emphasize that the same formula can be open to many different interpretations and accordingly support different business processes. By and large, we present a comprehensive and robust mathematical foundation on which many particular methods and business processes can be built. Trying to combine both tasks in one article would be rather impractical, given the amount of presented material. However, in the future, we are planning to address more particular business processes, exploited by certain financial institutions, based on this framework.

1. Defining contribution

We will start from contribution. The reason for this is that some works raise the question, if weights of securities within the investment portfolio should be defined relative to their beginning market value, or relative to the ending market value. Suppose that portfolio consists of *N* groups of financial instruments (it can be separate securities, industry sectors, etc.). Then, according to a definition of rate of return, we have the following.

$$R_p = \frac{E_p - B_p}{B_p} = \frac{\sum_{i=1}^{i=N} B_i (1+R_i) - \sum_{i=1}^{i=N} B_i}{B_p} = \frac{\sum_{i=1}^{i=N} \left[B_i (1+R_i) - B_i \right]}{B_p} = \sum_{i=1}^{i=N} \frac{B_i R_i}{B_p} = \sum_{i=1}^{i=N} W_i R_i \qquad (1)$$

where R_p is the portfolio rate of return; R_i is the group's rate of return; W_i is the group's weight within the whole portfolio; *i* - index denoting the group's number; index *p* relates to portfolio; *B* denotes the beginning market value; *E* is the ending market value.

Since formula (1) uses simple rate of return, in a strict mathematical sense, it is valid in the absence of cash flows. However, if cash transactions are small compared to the beginning market value, it also produces reasonable approximate results. This formula proves that when composing the total rate of return of the portfolio or benchmark from the rates of return of included groups of securities, we should use weights based on the *beginning* market values. This consideration is valid for all transformations in case of arithmetic attribution models, including contribution to relative return. The last one is defined as $D = R_p - R_b$, where index "*b*" corresponds to a benchmark.

In case of geometric attribution, the situation is different. The rates of return are defined using (1), while the contribution to relative return *D* (the parameter defining the difference between the benchmark's and portfolio's rates of return on the group level), and excess return (sometimes this term is used to denote the difference in rates of return on the total level) are defined by the *ending* market values. According to the definition of the geometric contribution to relative return, we have.

$$D = \frac{1+r}{1+R} - 1 \qquad (2)$$

Values of *r* and *R* denote the rates of return of the same group of securities within the portfolio and benchmark accordingly. The introduced relationship is not an obvious one. We can rewrite it as follows.

$$D = \frac{r-R}{1+R} = \frac{\frac{e-B}{B} - \frac{E-B}{B}}{1+\frac{E-B}{B}} = \frac{e-B-E+B}{B+E-B} = \frac{e-E}{E} = \frac{e}{E} - 1 \qquad (3)$$

where e and E are the ending market values of the portfolio's and benchmark's groups; B is the beginning market value that is chosen to be the same for the portfolio and benchmark (we can do this based on the definition of rate of return). Since we use a simple rate of return, as in (1), formula (3) is valid in the absence of cash transactions.

So, in case of geometric attribution, the contribution to relative return is completely defined by the ending market values.

2. Present arithmetic attribution models

The following notations will be in use. Capital letters will refer to a benchmark, while the lower-case letters will refer to a portfolio. Rate of return will be accordingly R and r, weights W and v, differences between the rates of return and weights are defined as follows:
$d_v = v - W$, $d_r = r - R$.

BHB arithmetic attribution model

The difference between the contribution by some group of stocks into the portfolio rate of return, and contribution of the same group into the benchmark's rate of return, is defined as follows.

$$\begin{aligned} D = rv - RW = (R + d_r)(W + d_v) - RW = Rd_v + Wd_r + d_v d_r = \\ = R(v - W) + W(r - R) + (r - R)(v - W) \end{aligned} \quad (4)$$

where notations for simplicity relate to one group of stocks, so there are no indexes in this formula.

The first term on the second line in expression (2) is a known *industry selection or timing*, the second term is the *stock selection*. We also have the third term in the BHB model, which is called *interaction*. In fact, this term represents the model's noise factor and, given the results presented in this article below, has no business meaning. There were a few attempts made to give a business interpretation to the interaction effect, but all of them are different and not really convincing. They seemed to have appeared due to our human ability to interpret whatever we encounter. In (Spaulding, 2003), the author acknowledges: “You may notice that I do not comment on the “other” effect, since there’s no intuitive way to anticipate what the result will be”, where “other” is used for denoting an interaction effect. The existence of interaction imposes limitations to the usage of two other attribution parameters. Ideally, as long as we factor the difference in rates of return between the benchmark and portfolio into the stock selection and industry selection, we would like for these components to constitute the *whole* difference. The presence of interaction jeopardizes the factoring and makes the attribution parameters less valuable or maybe worthless. A numerical example in Table 2 demonstrates this consideration, showing how the value of interaction can be comparable to the industry selection and stock selection values.

Table 1. Sample portfolio with four groups of stocks.

Group No.	R	r	W	v
1	0.4	0.1	0.2	0.3
2	0.3	0.1	0.3	0.25
3	0.15	0.45	0.35	0.25
4	0.2	0.3	0.15	0.2
Total	**0.2525**	**0.2275**	**1.0**	**1.0**

Table 2. BHB model. Attribution parameters for the input data from Table 1.

Group No.	Industry Sel	Stock Sel.	Interaction
1	0.04	-0.06	-0.03
2	-0.015	-0.06	0.01
3	-0.015	0.105	-0.03
4	0.01	0.015	0.005
Total	**0.02**	**0.0**	**-0.045**

Another important feature of the BHB model is the use of benchmark’s parameters as primary values, compared to portfolio’s parameters. Legally, portfolio and benchmark are equal entities. So, there are no any reasons to give any preference to a benchmark. Equal entities should be treated equally to preserve the balance. However, this is not the case for the BHB model. Let us explain this comment in more detail. If we take a look at definition of stock selection and industry selection in (4), we can see that *three* parameters from *five* belong to a benchmark, while only two parameters belong to a portfolio. This appearance asymmetry, in fact, leads to functional asymmetry, giving higher weight to benchmark's parameters. Numerical results will show this later. This potentially leads to stronger influence of benchmark parameters to the results of attribution analysis.

Brinson-Fachler's attribution model

The formula that is usually attributed to Brinson-Fachler's attribution model differs from the BHB model in definition of industry selection, which is defined as follows in our notations.

$$I = (R - B)(v - W) \tag{5}$$

where B denotes the total benchmark's rate of return.

Stock selection is the same, that is

$$S = W(r - R) \tag{6}$$

We can write the contribution to relative return for a single group in the following form.

$$\begin{aligned} D &= vr - WR = (W + d_v)(R + d_r) - WR = \\ &= W(r - R) + R(v - W) + (v - W)(r - R) = \\ &= W(r - R) + R(v - W) - B(v - W) + (v - W)(r - R) + B(v - W) = \\ &= W(r - R) + (R - B)(v - W) + (v - W)(r - R) + B(v - W) \end{aligned} \tag{7}$$

where variable B denotes the overall benchmark return.

So, we can see that in the Brinson-Fachler's model the industry selection $(R-B)(v-W)$ is represented by the second term (on the last line), from the total four that compose the contribution to relative return. Other terms are familiar stock selection $W(r-R)$, interaction $(v-W)(r-R)$, and a new summand $B(v-W)$, which beside the interaction represents another model's noise factor. Similar to the interaction term, we can try making some business sense of it, but this would not really be convincing and doubtfully have any practical value. Neither this term, nor the interaction term, are accepted by the community as legitimate attribution parameters. Unlike the interaction, the value of this term is always zero on the total (portfolio) level. Table 3 presents attribution parameters for the same data from Table 1.

Table 3. Attribution parameters computed using the Brinson-Fachler model for the data from Table 1.

Group No.	Industry Sel	Stock Sel.	Interaction
1	0.01475	-0.06	-0.03
2	-0.002375	-0.06	0.01
3	0.01025	0.105	-0.03
4	-0.002625	0.015	0.005
Total	**0.02**	**0.0**	**-0.045**

With regard to appearance asymmetry, this model includes *five* benchmark parameters from total *seven*, and *two* portfolio's parameters.

3. New scientific principles and validation criteria for developing attribution models

We cannot evaluate how objective are the attribution parameters found by BHB or Brinson-Fachler's methods in the above numerical examples. We just have to take the results for granted. At the same time, the large value of interaction indicates that the attribution parameters may be invalid. So, we need some tools and criteria to evaluate objectivity of attribution models.

Previously, we noticed the appearance asymmetry of formulas defining attribution parameters. On the other hand, benchmark and portfolio legally are equally important entities. This assumes the necessity of appearance symmetry of attribution formulas. The formulas must be such, that it should not be different if we compare benchmark to the portfolio, or portfolio to the benchmark. The results can differ in algebraic signs, but not in absolute values. Besides, we should also be able to compare two benchmarks, or two portfolios.

Appearance and functional symmetries

So, attribution methods have to preserve the *appearance symmetry*, which assumes at least the same number of parameters corresponding to portfolio and benchmark. Secondly, formulas must have *functional symmetry* as well. It means that if we substitute the benchmark's data by portfolio's data, and vice versa, then the results can differ in algebraic signs, but they have to be the same in absolute values. We will call this operation and validation principle as a *data swapping symmetry*.

Independence of attribution parameters

When we factor contribution to relative return into the stock selection and industry selection, we would like to obtain *independent* parameters. The analogy can be two-dimensional plane. The location of each point on this plane is defined by two independent values, which are coordinates of this point on the abscissa and ordinate. With this regard, we cannot guarantee that the stock selection and industry selection defined by the BHB and Brinson-Fachler's models are independent. In fact, from the mathematical perspective, which we discuss below, they are not.

Algebra has well developed formal apparatus to define such independent values for mathematical expressions composed of combination of variables. The central idea is the introduction of *canonical* forms of such mathematical expressions. Sometimes the canonical form can be found by simple algebraic rearrangements of the initial expressions, and we will use this approach as one of the methods. The other approach is to use matrix algebra and find the eigenvalues of the appropriate matrix whose elements are composed from the terms of original mathematical expression. We will use this method too. Note that both approaches are *independent* mathematical methods. So, if we manage to obtain the same result by both methods, this will be a strong proof of the result's validity. Besides, if we manage to factor the contribution to relative return by these methods, it will *necessarily* guarantee the *independence* and *uniqueness* of such obtained parameters, and the absence of any other associated values which we considered as noise factors in the previous models. Whatever value of contribution to relative return we will have, it will be *always* factored into two, and only two, *independent* values without any interaction or other noise terms. If we take a look at formulas (4) and (7), we can see that they do not represent the canonical forms of contribution to relative return because of the presence of noise factors.

So, the existence of a *canonical form* of contribution to relative return should be introduced as the next principle in developing of attribution methods, which guarantees the independence and uniqueness of attribution parameters. However, this is not always possible to prove because of mathematical complexity of characteristic equation, when its solution cannot be found. In such a situation, the heuristic approach can help, while the rest of principles and validation criteria of the development framework help to validate such a heuristically discovered method.

A symmetrical data set

Another approach to validate attribution models is to use a symmetrical data set. If we compose such portfolio and benchmark that the weights and rates of return are symmetrical, then attribution parameters have to be symmetrical as well. The difference between the discussed earlier principle of data swapping symmetry and a symmetrical data set test is that in the last case we have to do only one set of computations, while data swapping validation requires two sets. Of course, the data swapping validation can be applied to a symmetrical data set, although this is a particular case of a more general principle of data swapping symmetry.

Table 4 presents an example of a symmetrical data set, in the previous notations, while Table 5 and Table 6 show attribution parameters calculated for this data set using the BHB and Brinson-Fachler's models.

Table 4. Sample symmetrical data set

Group No.	R	r	W	v
	0.3	0.2	0.2	0.3
	0.3	0.2	0.3	0.2
	0.2	0.3	0.3	0.2
	0.2	0.3	0.2	0.3
Total	**0.25**	**0.25**	**1.0**	**1.0**

Table 5. Attribution parameters for the symmetrical data set from Table 4, calculated using the BHB model.

Group No.	Industry Sel	Stock Sel.	Interaction
1	0.03	-0.02	-0.01
2	-0.03	-0.03	0.01
3	-0.02	0.03	-0.01
4	0.02	0.02	0.01
Total	**0.0**	**0.0**	**0.0**

Table 6. Attribution parameters for the symmetrical data set computed by Brinson-Fachler's model.

Group No.	Industry Sel	Stock Sel.	Interaction
1	0.005	-0.02	0.015
2	-0.005	-0.03	-0.015
3	0.005	0.03	-0.035
4	-0.005	0.02	0.035
Total	**0.0**	**0.0**	**0.0**

We can see that both models produce asymmetrical attribution parameters. So, they do not sustain the validation by a symmetrical data set.

Ratio validation criterion

This validation criterion has been discovered *after* we developed several new attribution models based on introduced conceptual framework. So, to some extent, the discovery of this validation criterion can serve as a proof of validity of the whole conceptual framework and these new methods. The idea behind this criterion is the following. Suppose we evaluate attribution parameters by two methods and obtain two sets of attribution parameters. Let it be the industry selection and stock selection obtained by some arithmetic and geometric attribution models. Then, if the methods are correct, it is reasonable to expect that the ratio of attribution parameters (industry selection value divided by the stock selection) should be the same for different methods. This consideration has to be valid both on the total level and on the group level. In a symbolic form, it can be written as follows.

$$\frac{I_T^G}{S_T^G} = \frac{I_T^A}{S_T^A}\,; \quad \frac{I_j^G}{S_j^G} = \frac{I_j^A}{S_j^A} \qquad (8)$$

where indexes G and A relate to geometric and arithmetic attribution models accordingly; index T denotes the total values; index j denotes the j-th group of financial instruments.

So, we can introduce the *ratio validation criterion* as follows: "*The ratio of the appropriate attribution parameters for the same input data should be identical across different attribution methods designed to process this type of data within the same comparison context.*"

We have to stress that *context* of compared methods should be the same. For instance, we can compare the arithmetic attribution method, considering the *whole* value of contribution to relative return, to the geometric attribution method that also factors into similar attribution parameters the *whole* value of the geometric contribution to relative return.

However, if another method adjusts the rate of return for some threshold value, then the context of methods becomes different, and the ratio validation criterion cannot be applied.

4. Developing a new arithmetic attribution model

Now, as we introduced the general principles and validation criterion of the new conceptual framework, we will develop new attribution methods to illustrate the application of declared principles. At first, we find the canonical form of a contribution to relative return for the arithmetic attribution model. We can rewrite (4) as follows.

$$\begin{aligned} &D = R(v-W) + W(r-R) + (r-R)(v-W) = \\ &R(v-W) + W(r-R) + rv - rW - Rv + RW = R(v-W) + v(r-R) \end{aligned} \qquad (9)$$

This formula is similar to what we have seen in the BHB model. The difference is that we use the weight from a portfolio, not from a benchmark. Alternatively, we can regroup terms in (4) as follows.

$$D = W(r-R) + r(v-W) \qquad (10)$$

Now, we can express value D as a half sum of two expressions using (9) and (10). The result will be as follows.

$$D = \frac{(r+R)}{2}(v-W) + \frac{(v+W)}{2}(r-R) \qquad (11)$$

Formula (11) presents the mathematical canonical form of a contribution to relative return. It includes only two attribution parameters, which are independent as functions of arguments $(v-W)$ and $(r-R)$, which we consider as independent variables for the purposes of attribution analysis. (Reviewer of the first version of this article suggested another wit derivation of (11). So, there are several ways to obtain the same result.) The first term in (11) should be interpreted as a new industry selection, while the second term is a new stock selection. For certainty, we can call these parameters as a *symmetrical* industry selection and a *symmetrical* stock selection. The reason is that these parameters satisfy to symmetry requirements introduced earlier. The model itself will be called accordingly as a symmetrical arithmetic attribution model (SAA). The attribution parameters are defined as follows.

$$I_s = \frac{(r+R)}{2}(v-W) \tag{12}$$

$$S_s = (\frac{v+W}{2})(r-R) \tag{13}$$

where (12) defines a symmetrical industry selection, (13) defines a symmetrical stock selection.

The question that can be asked is as follows. Why these parameters are more objective than the ones produced by the BHB or Brinson-Fachler's attribution models? The answer contains conceptual, mathematical, business, and just common sense considerations. From the business perspective, in the new model, we do not have interaction effect or any other noise factors, never. So, unlike the case of the present models, whatever input data are, the contribution to relative return will be always decomposed into the sum of two independent meaningful parameters without any other noise factors. From strictly mathematical perspective, formula (11) represents a *canonical form* of the original mathematical expression (4) with regard to attribution parameters. This means that thus defined parameters are *independent* and *unique*, according to the properties of canonical forms for this type of mathematical expressions. On the opposite side, attribution parameters defined by the BHB or Brinson-Fachler's models are neither independent, nor unique (we alone introduced three different forms (4), (9), and (10) for the BHB model).

Let us evaluate the correspondence of this model to the conceptual framework.

Appearance and data swapping symmetry

Formula (11) includes three benchmark parameters and three portfolio parameters. So, the *appearance symmetry* is fulfilled, at least with regard to the number of parameters. We can see that if we swap the portfolio data and benchmark data, then the formula, in fact, remains the same. We just compare the benchmark to a portfolio in the same way as we compared the portfolio to a benchmark. So, the attribution parameters will be the same in absolute values, but different in algebraic signs. We can do the appropriate algebraic transformations substituting the benchmark's parameters by portfolio's parameters and vice versa, but for (11) this is an obvious consideration. This confirms the presence of the *data swapping symmetry* required by the conceptual framework. Table 7 presents these new attribution parameters for the data from Table 1. If we swap the benchmark's and portfolio's data, the absolute value of attribution parameters will remain the same, but the algebraic signs will change to opposite. So, both the analytical analysis and numerical example confirm the model's data swapping symmetry.

Table 7. Attribution parameters found using the SAA model. Data set from Table 1.

Group No.	Industry Selec.	Stock Selec.	Interaction
1	0.025	-0.075	0
2	-0.01	-0.055	0
3	-0.03	0.09	0
4	0.0125	0.0175	0
Total	**-0.0025**	**-0.0225**	**0**

Application of (11) to a symmetrical data set from Table 4 produces the attribution parameters presented in Table 8.

Table 8. Attribution parameters for the symmetrical data set calculated using the symmetrical arithmetic attribution model (SAA).

Group No.	Industry Sel	Stock Sel.	Interaction
1	0.025	-0.025	0

2	-0.025	-0.025	0
3	-0.025	0.025	0
4	0.025	0.025	0
Total	0.0	0.0	0.0

We can see that the attribution parameters are symmetrical, as it is required by the introduced conceptual framework. So, the model passed the test by a symmetrical data set as well.

5. Finding a canonical form of arithmetic contribution to relative return using eigenvalues

It is well known from linear algebra that linear forms can be transformed into diagonal ones using their eigenvectors. If we interpret this fact in a more practical form, it means that we can create a space of new *independent* informative values from the set of values given in a non-diagonal form. We can present the industry selection, stock selection and interaction for the arithmetic contribution to relative return from (4) in the following matrix form (we use the earlier notations).

$$\begin{Vmatrix} R(v-W)+p & p \\ p & W(r-R)+p \end{Vmatrix} \qquad (14)$$

What we did here is the following. The half of interaction value *p* is added to the industry and stock selections located on the main matrix diagonal. The matrix trace will be equal to contribution to relative return, while the second diagonal has been filled with halves of interaction value. Thus constructed matrix will always produce zero interaction value (assuming we can find its eigenvalues). Adding elements on the second diagonal pursues the idea to redistribute the interaction between eigenvalues. Solving characteristic polynomial of this matrix, we find the following eigenvalues.

$$\lambda_1 = \frac{1}{2}(rv - RW + \left[(I-S)^2 + 4p^2\right]^{\frac{1}{2}}) \qquad (15)$$

$$\lambda_2 = \frac{1}{2}(rv - RW - \left[(I-S)^2 + 4p^2\right]^{\frac{1}{2}}) \qquad (16)$$

where $I = R(v - W)$, $S = W(r - R)$.

Note that discriminant in these formulas is *always* non-negative value, so that these formulas are valid for any combination of rates of return and weights.

The reader can think of eigenvalues (15) and (16) as a mathematically correct way to redistribute the values of non-diagonal terms (together composing interaction) to the diagonal elements representing the industry selection and stock selection accordingly, added with a half of interaction value. We can also check compatibility with the BHB model scrutinizing eigenvalues. For example, if $d_r = 0$, then the first eigenvalue produces parameter similar to industry selection, while the second eigenvalue produces the stock selection, when $d_v = 0$. That is

$$\lambda_1 = \frac{(r + R)}{2}(v - W) \qquad (17)$$

$$\lambda_2 = \frac{(v + W)}{2}(r - R) \qquad (18)$$

Note that, in fact, we obtained exactly the same attribution parameters as in (12), (13), although this time we did this using independent approach. Values (17), (18) are guaranteed to be *independent* and *unique*, because they represent a *mathematical canonical form* of the contribution to relative return.

6. A new referential arithmetic attribution model

Previously, we showed that the Brinson-Fachler's attribution model does not satisfy to data swapping symmetry principle, and does not support the symmetry of attribution parameters for the symmetrical input data set. Here, we introduce a new attribution model implementing the idea of some reference value, as the Brinson-Fachler's model does referring to the total benchmark's rate of return. However, we will not define this value for now. Let us denote it as some threshold value *T*. We will use the following substitutions: $r_T = r - T$, $R_T = R - T$, that is the rates of return adjusted for the reference value *T*. As a particular case, value *T* can be calculated as an arithmetic average of the portfolio's and benchmark's total rates of return. We will define this new referential arithmetic attribution model (RAA) as follows.

$$I_s^T = \frac{(r_T + R_T)}{2}(v - W) \tag{19}$$

$$S_s^T = \frac{(v + W)}{2}(r_T - R_T) = \frac{(v + W)}{2}(r - R) \tag{20}$$

where I_s^T and S_s^T are new industry selection and stock selection adjusted for a threshold value.

Thus, the defined attribution model does not have "noise" factors. The total contribution to relative return consists of only industry selection and stock selection. It can be proved as follows.

$$\begin{aligned} S_s^T + I_s^T &= \frac{(r_T + R_T)}{2}(v - W) + \frac{(v + W)}{2}(r_T - R_T) = \\ &= \frac{1}{2}\left(2r_T v - 2R_T W\right) = r_T v - R_T W = D_T \end{aligned} \tag{21}$$

This formula discovers that the notion of contribution to relative return D_T has to be applied to the sector's rate of return that is adjusted for the reference value. Earlier, we used the contribution to relative return defined as $D = rv - RW$. Suppose, we would like to keep this definition, which is reasonable assumption, because this way we preserve the consistency of the overall approach and acquire one more connection with the proven method. Then, the equality $D = D_T$ has to be valid, that is

$$(r - T)v - (R - T)W = rv - RW \tag{22}$$

Solutions of this equation are as follows. One is $v = W$ for all $T \neq 0$. The second solution is $T = 0$. It is valid for any values of the sectors' weights.

The first solution does not have business sense to us, given the nature of the problem. The second solution is an interesting development. In fact, this particular case is a symmetrical arithmetic attribution model (SAA) that we introduced previously. Such a conversion of one model into another on the boundaries of parameters' domain is one more proof of both models' validity, because we developed them independently based only on the general principle of symmetry.

The definition of contribution to relative return for the adjusted rates of return can be introduced as follows. Comparison of rates of return relative to some threshold value means that we should weight the rates of return relative to this threshold. Otherwise, there is no way to weight the threshold value proportionally to the sector's weight. This confirms the validity of introducing contribution to relative return as a parameter that is adjusted for the threshold value, using the original weights of groups.

The attribution model (19), (20) preserves the original idea to compute attribution parameters relative to some reference value. This model is more objective in terms of factoring the contribution to relative return into two parameters. This is because it confirms to symmetry principles, and it does not have noise factors such as interaction and other terms.

Let us find the reference value. It turns out that the group level attribution parameters satisfy to symmetry requirement only if the reference value *T* is such that it does not change when we swapped the data. Otherwise, the attribution parameters for the original and swapped data are asymmetrical.

For example, this condition fulfills when the reference value *T* is proportional to arithmetic average of the portfolio's and benchmark's returns. If we denote the total benchmark's rate of return as *B* and the total portfolio's rate of return as *P*, then the reference value *T* in the RAA model can be defined as follows.

$$T = C \times \frac{B + P}{2} \qquad (23)$$

where C is a real number.

If we denote the reference value as a function of B and P, that is $T = T(B, P)$, then any value of this function can be used as a reference value T, provided the function satisfies the following condition.

$$T(B, P) = T(P, B) \tag{24}$$

In the numerical examples below, we use the reference value of $T = \frac{(B+P)}{2}$. Table 9 represents attribution parameters for the symmetrical data set, when the referential arithmetic model is used. Results demonstrate that these attribution parameters are symmetrical. This is an expected result given the properties of the RAA model that should produce symmetrical attribution parameters for the symmetrical input data.

Table 9. Attribution parameters for the symmetrical data set when the RAA model is applied.

Group No.	Industry Sel	Stock Sel.	Interaction
1	0	-0.025	0
2	0	-0.025	0
3	0	0.025	0
4	0	0.025	0
Total	**0.0**	**0.0**	**0.0**

The next test should be the appearance symmetry and the data swapping symmetry. We can see that the appearance symmetry is preserved for (19) and (20) when we define the threshold value as in (23) or, in a more general case, as in (24). Data swapping symmetry can be proved analytically rewriting (19) and (20) for the swapped portfolio's and benchmark's data. Tables 10 and 11 present numerical examples confirming the property of data swapping symmetry for this model, when we use the data set from Table 1. We can see that the attribution parameters are the same in absolute values, but have the opposite algebraic signs.

Table 10. Attribution parameters for the symmetrical referential arithmetic attribution model. Data set from Table 1.

Group No.	Industry Sel	Stock Sel.	Interaction
1	0.001	-0.075	0
2	0.002	-0.055	0
3	-0.006	0.09	0
4	0.0005	0.0175	0
Total	**-0.0025**	**-0.0225**	**0.0**

Table 11. Attribution parameters for the same swapped data.

Group No.	Industry Sel	Stock Sel.	Interaction
1	-0.001	0.075	0
2	-0.002	0.055	0
3	0.006	-0.09	0
4	-0.0005	-0.0175	0
Total	**0.025**	**0.0225**	**0.0**

The following considerations should be taken into account when defining the value of threshold. We discussed before that the absolute values of attribution parameters cannot depend on the direction of comparison. The reason is that these parameters objectively present the *difference* between two entities of the same type. When we introduced the reference value *T*, we changed the compared values. This change of compared values has to be balanced in such a way that the data swapping does not change the reference value, and the appearance symmetry has to be valid too.

We also should consider the requirement that the attribution model mathematically has to be presented by a canonical form. Formulas (19), (20) can be considered as a canonical form of the contribution to relative return adjusted for the threshold value *T* with regard to independent variables $(v-W)$ and $(r-R)$, provided their coefficients do not depend on these variables. When we choose the threshold values according to (23) or (24), this condition holds true. So, (19), (20) represent a mathematical canonical form.

7. Ratio validation criteria

For this purpose, we compare attribution parameters for the symmetric arithmetic attribution model and symmetric geometrical attribution model (SGA) developed in [4], and also described in [5]. We use the same data from Table 1 for both models. Note that these models have the same attribution context. Namely, both models factor the *whole* contribution to relative return into two attribution parameters. We calculated the ratio of stock selection to industry selection on the total and group levels for these models. The results are presented in Table 12. We can see that they are identical. So, both models confirm to the ratio validation criteria, that validates both models.

Table 12. Application of ratio validation criterion to SAA and SGA attribution models.

Group No.	Ratio of Attrib. parameters' for SAA	Ratio of Attrib. parameters' ratio for SGA
1	-3.0	-3.0
2	5.5	5.5
3	-3.0	-3.0
4	1.4	1.4
Total	**9.0**	**9.0**

Conclusion

The introduced conceptual framework for developing attribution models proved to be a valuable and practical instrument, which includes a set of principles and validation tools to which the developed attribution models should confirm. We analyzed two existing attribution models and discovered that they do not satisfy some requirements from the perspective of this new development framework. We introduced two new attribution models that serve as the illustration how the new concepts can be applied in practice, completing every step from the models' inception and through all validation and testing phases. Further, we compared two different models in order to demonstrate the application of a ratio validation criterion.

This is a special validation criterion. Unlike the previous validation tools, this one is used for validation across different models.

Lack of space did not allow presenting more pertinent issues, such as, for instance, relationship of introduced concepts and methods with absolute profit and loss, which could be an interesting development too. However, since the only basis of proposed framework is rates of return, such relationships can be exposed fairly straightforwardly once we know some absolute monetary value, such as the beginning market value. Another interesting aspect is linking of attribution parameters, of which some results can be found in (Shestopaloff, 2009a).

We think that overall the results convincingly demonstrate the robustness and objectivity of the proposed conceptual framework for developing attribution models, and can be successfully applied to development of other attribution models. Although we restricted ourselves to analysis of equity portfolios, the same principles are transferable to the attribution analysis of fixed income securities.

The summary of introduced principles and validation tools is presented below in a tabular form below.

1.	A correct comparison method has a property of symmetry, which assumes that the absolute value of the result of comparison should not depend on the direction of comparison, whether the first entity is compared to the second, or vice versa, although algebraic signs of the results of comparison can be opposite. In attribution analysis, this principle leads to a *data swapping symmetry* when swapping of the benchmark's and portfolio's data has to produce the same absolute values of attribution parameters, while the algebraic signs are opposite.
2.	The data swapping symmetry results in *"appearance" symmetry* of mathematical formulas defining attribution parameters.
3.	Existence and uniqueness of a canonical form of the contribution to relative return.
4.	Attribution parameters calculated for the symmetrical data set have to be symmetrical too.
5.	Ratio validation criterion assumes that the ratios of attribution parameters on the

	total and group levels have to be the same for different attribution methods developed within the same context.

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