HIGH ACCURACY APPROXIMATION ANALYTICAL METHODS FOR CALCULATING INTERNAL RATE OF RETURN

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Diversity of methods for calculating rates of return on investments has been propelled by a combination of ever more sophisticated business needs on one side and limitations of the available computing power on the other. With current advances in technology the next logical step would be to unify and prioritize the multitude of available methods into an optimized set of quantitative algorithms.

This study explores internal rate of return. High accuracy approximation method for calculating internal rate of return is suggested. It is also demonstrated that dollar weighted and time weighted methods closely relate to each other with the time weighted method being just a particular case of dollar weighted rate of return when considering linear approximation of internal rate of return equation.

Introduction

Internal rate of return (IRR) is arguably one of the most objective ways of evaluating the rate of return on investment. However there are disadvantages. Two features are usually emphasized:

- First, it is computationally intensive method. Presently it shouldn't be considered as an issue given the increased computational power of basic PCs.
- Second argument concerns the cash flows allegedly masking "pure" rate of return. There are pros and cons in favour of this argument. Some considerations will be presented later.

In this article we would like to introduce a relatively simple analytical approximation method for calculating internal rate of return with high levels of accuracy. This method is based on Taylor expansion for non-linear members of IRR equation. The accuracy typically is 10^{-4} or better when using the basic calculation procedure. However, the required accuracy can be controlled and enhanced even further via slightly modified approach also presented in the article. Thus found value can be considered as true IRR for all imaginable practical purposes. Both discrete and continuous compounding cases are analyzed. As an enhancement, the methods using quadratic terms of Taylor expansion are considered too. These methods provide even higher accuracy. The price is more complicated formulae.

In this article we also demonstrate that approximation of IRR equation using linear Taylor expansion at point with zero rate of return results in a Modified Dietz formula both for discrete and continuous compounding. It means that separation of performance measurement methods onto dollar weighted and time weighted rates of return is somewhat artificial. In fact, time weighted rate of return is derived from dollar weighted rate of return as a particular approximation case.

Internal Rate of Return

IRR can be found using equations for continuous compounding as follows [see Spaulding, 1997].

$$B+\sum_{j=1}^{N} C_j e^{-Rt_j} - Ee^{-Rt_{N+1}} = 0 \qquad (1)$$

where B – beginning market value; E – ending market value; C_j – cash flow; t_j – time from beginning of period until cash flow occurred or length of the overall period (t_{N+1}) measured in units of chosen atomic period; R – IRR to be found; e – exponent (e=2.71…).

For discrete compounding

$$B+\sum_{j=1}^{N} C_j (1+R)^{-t_j} - E(1+R)^{-t_{N+1}} = 0 \qquad (2)$$

The equivalent form of equations (1) and (2) can be derived by multiplying both parts of each equation by $e^{Rt_{N+1}}$ and $(1+R)^{t_{N+1}}$ accordingly. For example, (2) will be rewritten as follows:

$$E = B(1+R)^{T_{N+1}} + \sum_{j=1}^{N} C_j (1+R)^{Tj} \qquad (3)$$

where Tj is now time from when cash flow occurred till the end of the period measured in units of chosen atomic period. T_{N+1} is the length of the overall period. Time is measured in units of chosen atomic period.

Relationship of Period Length and ROR

Rate of return *R* relates to atomic period one wants to find ROR for. An *Atomic Period* is any fixed period of time used as an agreed upon unit of time for specific set of calculations. Generally speaking, the period can be of any arbitrary length.

It is important to remember that R found from (1) - (3) is the ROR for a chosen atomic period though formulas (1) – (3) are applied to a longer or shorter overall period. It is not a ROR for the overall considered period. For example, atomic period is a quarter. Overall period is three quarters. Then R calculated on the base of (1) - (3) relates to quarter rate of return R_q. ROR for the overall period three quarters will be $R_{tot} = (1+R_q)^3 - 1$. For the whole year it will be $R_{yearly} = (1+R_q)^4 - 1$. Alternatively, atomic period can be chosen as having three quarters length.

In general case if ROR for atomic period is R and the length of period is L, then ROR R_2 for period with another length L_2 can be found as

$$R_2 = (1+R)^{\frac{L_2}{L}} - 1 \qquad (4)$$

Formula (4) can be used for estimation of error introduced by geometrical linking compared to IRR and ROR calculated using other methods. That is, find geometrical linking ROR and then roll back it to smaller periods calculated by other methods. The opposite procedure can be done as well – from shorter period to a longer one.

In further consideration we assume that considered overall period is equal to atomic period, that is $T_{N+1} = 1$. This assumption is common across other methods for ROR calculation. Generalization of such achieved result to arbitrary length period can be done using (4).

It is also possible to solve equations (1) - (3) directly when reporting period consists of atomic periods using same numerical methods. Relative to suggested approach it means necessity to expand also the first term $B(1+R)^{T_{N+1}}$ on the right side of (3). This results in lower accuracy of solution compared to true IRR value. So, in order to retain accuracy, the high order terms have to be used in Taylor expansion, like quadratic terms for example.

Geometric Linking

Geometric linking and its modification, linked IRR method, pretend to provide “pure” rate of return. However, it should be used with proper interpretation because geometrical linking disregards the difference how much has been invested during particular period. For example, one invested $10 at the beginning of the first period. At the end of this period market value became $19. So, rate of return for this period is 90 %. Next period we invested additional $100. This investment grew to $129 at the end of the second period. Rate of return is 8.4 %. Geometrical linking produces 106 % total return while actual income is $19 gained from total investment $110. Intuitively number 106 % doesn’t look right.

Taylor Series

The approach suggested in this article is based on Taylor series expansion. Taylor series is a form of approximate presentation of function in the vicinity of particular point [see Max Kurtz, 1991].

For example, Modified Dietz formula can be easily derived from (3) or (2) using Taylor expansion at point R=0.

Approximation Analytical Method for IRR

This paragraph introduces proposed approximation method. The idea behind method is to approximate non-linear terms in the sum (3) by Taylor series using linear or linear and quadratic terms. First solution is found using Taylor expansion at point R=0. Thus found solution $R=R_0$ is used then as a point for Taylor expansion. Using Taylor expansion at R=0 leads to:

$$E = B(1+R_0) + \sum [C_j + C_j T_j R_0] \qquad (5)$$

Solution of this equation is as follows

$$R_0 = \frac{E - B - \sum C_j}{B + \sum C_j T_j} \qquad (6)$$

It happened to be Modified Dietz formula (as expected).

Equation (3) rewritten using Taylor expansion with linear term only looks as follows:

$$E = B(1+R) + \sum_{j=1}^{N} C_j [(1+R_0)^{T_j} + T_j (1+R_0)^{Tj-1} (R - R_0)] \qquad (7)$$

where R is the approximate IRR to be found.

Solving equation (7) results in the following expression for approximate value of IRR.

$$R = \frac{E - B - \sum_{j=1}^{j=N} [C_j (1+R_0)^{T_j} - C_j T_j (1+R_0)^{T_j - 1} R_0]}{B + \sum_{j=1}^{j=N} C_j T_j (1+R_0)^{T_j - 1}} \qquad (8)$$

Following the same path as for discrete periods one can derive analog of formula (8) for continuous compounding. First iteration at point R=0 gives expression (6), that is again Modified Dietz formula. Taylor expansion at point $R=R_0$ results in the following equation for ROR calculation:

$$R = \frac{E - B - \sum_{j=1}^{j=N} C_j e^{T_j R_0} [1 - T_j R_0]}{B + \sum_{j=1}^{j=N} C_j T_j e^{T_j R_0}} \qquad (9)$$

Precision of suggested method

Next question is how accurate the estimation (8) is relative to true IRR. This can be done in two ways. First, using (8) for a representative set of practical scenarios and finding the precision for these scenarios compared to true IRR. Then assume that all following results are not worse than the ones given by the representative set.

Second approach proposes to have direct control over the accuracy of calculation. To implement it one should substitute R instead of R_0 at (8) and find next iteration R_2 of ROR value. If absolute difference $|R_2 - R|$ is better than required precision, the calculation process completes. If not, the next iteration must be used and so forth until the required precision is achieved.

Direct use of formula (8) gave accuracy not worse than 10^{-4} for all simulated data. In order to explore extreme scenarios, the simulation data we used included data sets with RORs of up to 100%. Data sets were characterized by total value of cash flows relative to beginning market value (both absolute and algebraic sums); average rate of return for all periods (number of periods from 6 to 12); standard deviation of period rates of return. Calculated rates of return were compared against true IRR, Modified Dietz, geometric linking and ROR calculated using (10), (11) when R_0=0.

Some results of this comparison are in the Table 1 below.

Table 1. Rates of return calculated by different methods for the same input data.

Average period return	Standard Deviation of period returns	Cash Flow / Beg. Market Value	IRR (true internal rate of return)	IRR calculated using linear approxi-mation	Modified Dietz formula	Geometric linking	Using second term of Taylor expansion
0.09	0.03	1.12	0.82228	0.82226	0.79368	0.77166	0.83420
0.04	0.08	0.96	0.25169	0.25169	0.24873	0.26813	0.25205
0.12	0.11	0.09	1.20685	1.20685	1.20000	1.21221	1.21046
0.05	0.05	0.09	0.43188	0.43188	0.43077	0.43092	0.43210
0.21	0.20	-0.15	2.27982	2.27983	2.33333	2.46680	2.21831
0.07	0.04	-0.07	0.64333	0.64333	0.64553	0.64478	0.64268
0.10	0.11	0.31	0.82055	0.82055	0.81132	0.84868	0.82335
-0.14	0.23	0.96	-0.7208	-0.7215	-0.7665	-0.7242	-0.73808
-0.02	0.07	3.06	-0.23562	-0.23562	-0.24184	-0.12144	-0.23635
0.00	0.01	-0.80	-0.0753	-0.0753	-0.07444	0.03333	-0.07526
0	0	1.08	0	0	0	0	0
0	0	0.58	0	0	0	0	0

Table 2. Differences in rates of return for different methods compared to IRR (internal rate of return).

Average period return	Standard Deviation of period returns	Cash Flow / Beg. Market Value	IRR (true internal rate of return)	IRR minus IRR linear Approximation ROR	IRR minus Modified Dietz ROR	IRR minus Geometric linking ROR	IRR minus second term Taylor expansion ROR
0.09	0.03	1.12	0.82228	0.00002	0.02860	0.05062	-0.01192
0.04	0.08	0.96	0.25169	0.00000	0.00296	-0.01644	-0.00036
0.12	0.11	0.09	1.20685	0.00000	0.00685	-0.00536	-0.00361
0.05	0.05	0.09	0.43188	0.00000	0.00111	0.00096	-0.00022
0.21	0.20	-0.15	2.27982	-0.00001	-0.05351	-0.18698	0.06151
0.07	0.04	-0.07	0.64333	0.00000	-0.00220	-0.00145	0.00065
0.10	0.11	0.31	0.82055	0.00000	0.00923	-0.02813	-0.00281
-0.14	0.23	0.96	-0.72084	0.00070	0.04566	0.00338	0.01724
-0.02	0.07	3.06	-0.23563	0.00000	0.00621	-0.11418	-0.00073
0.00	0.01	-0.80	-0.0753	0.00000	0.00086	-0.04197	-0.0004
0	0	1.08	0	0	0	0	0
0	0	0.58	0	0	0	0	0

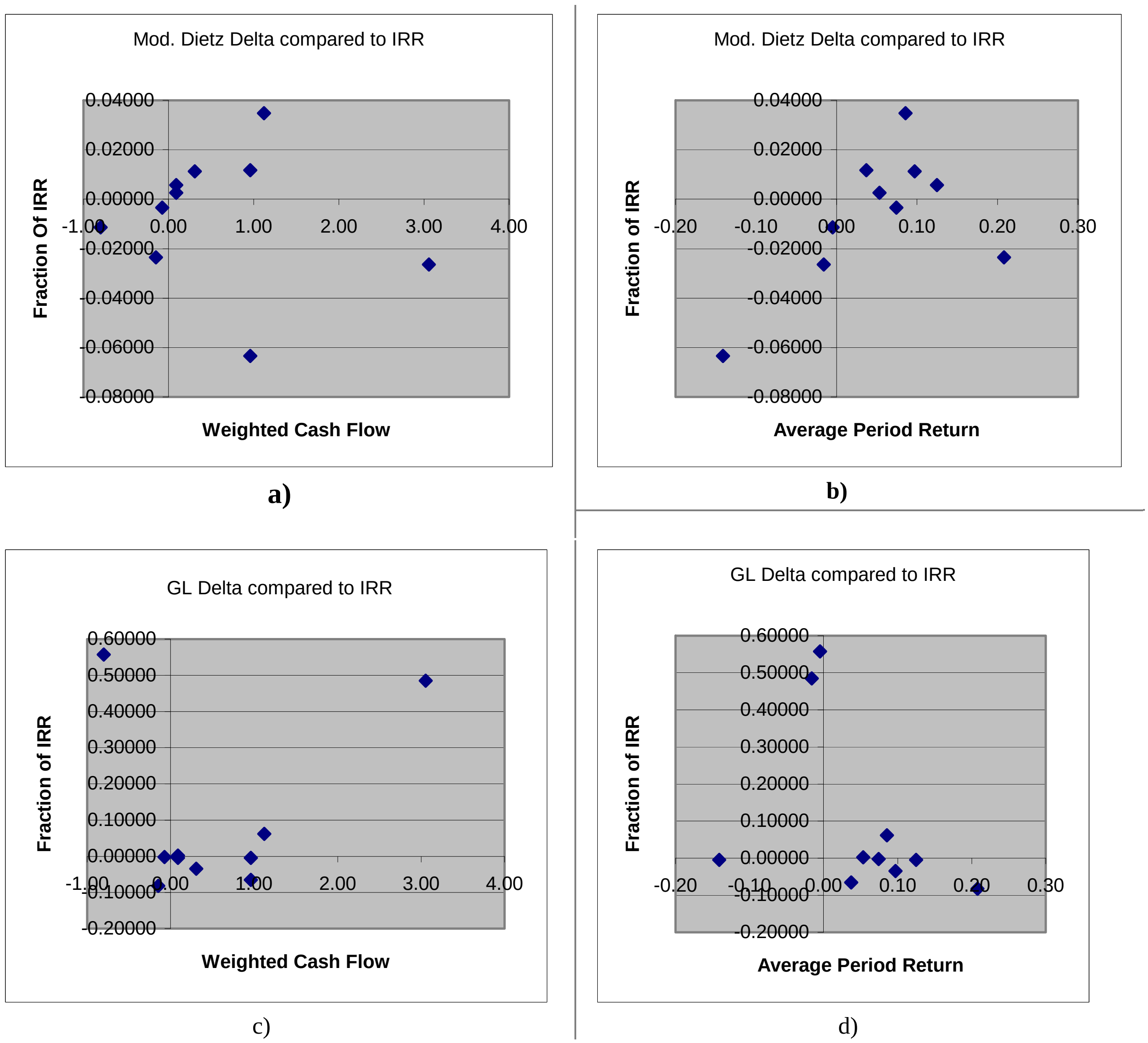

Fig. 1. Spread between internal rate of return (IRR) and the ones calculated using geometric linking (GL) and Modified Dietz methods. Parameter "Weighted Cash Flow" is a sum of cash flows divided by beginning market value; average period return is the mean of period returns. Delta means the difference between IRR and analyzed rate of return.

Discussion of numeric results

Analysis of table data and Fig. 1 highlights the following.

- The smaller the absolute value of rate of return, the smaller the discrepancy between rates of return calculated by different methods. The value of cash flow has negligible influence in this case.
- Geometric linking method produces the largest deflection comparing to IRR method. Modified Dietz method produces the differences of 5-15 times less for the same sets of input data.

- The accuracy of Modified Dietz method deteriorates with increase in both, cash flows and rate of return. This is a readily expected result given that Modified Dietz method is a linear approximation of IRR equation at point of zero rate of return.
- Weighted cash flow influences the accuracy of geometric linking method compared to IRR method. The larger the cash flows are relative to the beginning market value, the bigger is the difference between rates of return calculated by IRR and geometric linking methods.
- Increase in standard deviation of period returns and relative value of sum of absolute cash flows results in high discrepancy with internal rate of return. However, the algebraic sign of decrements is random.
- Proposed approximate methods for calculating internal rate of return provide very high accuracy for all practically meaningful scenarios.

<u>Using Linear and Quadratic Terms of Taylor expansion</u>

Adding quadratic term of Taylor expansion for non-linear members on the right side (3) results in the following quadratic equation for R.

$$R^2[\sum_{j=1}^{j=N}\frac{1}{2}C_jW_j(W_j-1)(1+R_0)^{W_j-1}]+R[B+\sum_{j=1}^{j=N}C_jW_j[(1+R_0)^{W_j-1}-R_0(W_j-1)(1+R_0)^{W_j-2}]]+$$
$$[B+\sum_{j=1}^{j=N}C_j[(1+R_0)^{W_j}-W_jR_0(1+R_0)^{W_j-1}+\frac{1}{2}R_0{}^2W_j(W_j-1)(1+R_0)^{W_j-2}]-E]=0 \quad (10)$$

The positive root of this quadratic equation is the searched value of R, that is

$$R=\frac{-b+\sqrt{b^2-4ac}}{2a} \quad (11)$$

where *a*, *b*, are equal accordingly to coefficients in (10) for R^2, *R*; *c* is a free member of this equation.

Similar quadratic equation for continuous compounding looks as follows:

$$R^2[\frac{1}{2}Be^{R_0}+\sum_{j=1}^{j=N}\frac{1}{2}C_je^{R_0W_j}W_j{}^2]+R[Be^{R_0}(1-R_0)+\sum_{j=1}^{j=N}C_jW_je^{R_0W_j}(1-R_0W_j)]+$$
$$[Be^{R_0}(1-R_0+\frac{1}{2}R_0{}^2)+\sum_{j=1}^{j=N}C_je^{R_0W_j}[1-R_0W_j+\frac{1}{2}W_j{}^2R_0{}^2]-E]=0 \quad (12)$$

Solution is also given by (11).

Discussion

Usage of formulas (8) and (9) and optionally formulas (10) – (12) implementing proposed approach should be considered in the following aspects:

- Business need
- Compliance with general standards and relevance to existing methods
- Acceptable level of complexity for average prospective consumers
- Sufficiency of computational resources
- Data feeds availability
- Applicability to future business requirements and business needs

Business need.
There is a definite business need in more precise and objective measurements of investment performance. It is stipulated by more demanding and skilful investors, greater publicity and data accessibility, more volatile markets, escalating competition, stricter regulations, diversification of business operations, etc. It is quite obvious that having good objective criteria for performance measurement would be extremely helpful to facilitate investment business, simplify reporting and streamline business operations. On a pre-trade side it could be very valuable too – valuating such objective criteria via modeling of different scenarios could certainly result in better investment decisions and practices.

Compliance with general standards and relevance to existing methods
IRR as a method and its extension such as linked IRR are already accepted by industry. Bringing simple methods for IRR calculation with high accuracy (like the ones suggested in the article) will provide wider usage of these methods. Second aspect: close relationship of IRR and other methods previously considered as stand alone entities creates a very good foundation for the following merge of some methods and better standardization of performance measurement business.

Acceptable level of complexity for average prospective users
Formulas (8) and (9) are fairly simple comparing to high-end financial mathematics used routinely in financial institutions today. Knowledge of high school calculus is sufficient to understand and use these formulae.

Sufficiency of computational resources
All formulae and computational performance were tested on a basic laptop using real and simulated data. The code has been written in C++. Cumulative operations were done via STL generic algorithms, such as accumulate(), inner_product(), etc. Specific operations were implemented using function objects passed to generic algorithms as parameters. Performance in terms of computational operations was very good. The difference between processing time for Modified Dietz and proposed method has been in the range 12-38 % depending on the data sets. So, computationally suggested methods do not require any special treatment.

Data feeds availability
Suggested methods do not use any additional input data compared to traditional methods.

Applicability to future business needs
Presently the following investments performance measurement trends can be observed.

- Demand for high accuracy objective characteristics.
- Diversification of portfolio management methods, strategies and instruments involved.
- Demand for up-to-date, close to real time performance measurement.
- Desire to evaluate efficiency of trading scenarios and solutions before trade.

Proposed methods do not contradict with any of the above requirements. They exploit consistent and predictable parameters sensitive to changes of the portfolio value and its components. Methods do not need any special data feeds restricting their applicability to different types of trading strategies and solutions or type of funds. Methods particularly are very accommodative to reporting periods of arbitrary length. It makes them valuable analysis tool both for post and pre-trade researches, as well as for real time performance measurement.

Conclusion

Proposed methods for calculating internal rate of return provide very high accuracy of calculation and can be considered as precise methods for all practical purposes. Methods are very stable and at the same time very sensitive to variations in market value and cash transactions of portfolio on the total portfolio, asset class and security levels.

It is demonstrated that time weighted rate of return such as Modified Dietz method is derived from dollar weighted rate of return, IRR method, as a particular approximation case. Thus there are no actually two different approaches – it's just an approximation of the same precise method. This lays down very strong case for unifying numerous performance measurements methods toward few standard ones. Eventually it will make performance measurement business more standard, objective and streamlined.

Presented research showed that internal rate of return should have a key role in the performance measurement being the parent of other used methods treated presently as independent criteria. IRR equation can be extended to cover multiple atomic periods within a single reporting period. Thus it can serve many more business scenarios. Given the computing power of modern enterprises, even small ones, it is not an issue to solve numerically such equations anymore. At the same time, geometric linking turns out to be not the universal criteria of "pure" rate of return. It has its own limitations. So, IRR should be given an adequate respect in hierarchy of methods used for ROR calculation.

Proposed methods could create foundation for the following unifying of performance evaluation standards based on accurate calculation of IRR using proposed approximation formulae and direct relationship of IRR with other standard methods.

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