CONSISTENT LINKING CONCEPT FOR FAST CALCULATION OF RATE OF RETURN AND RESEARCH OF INVESTMENT STRATEGIES

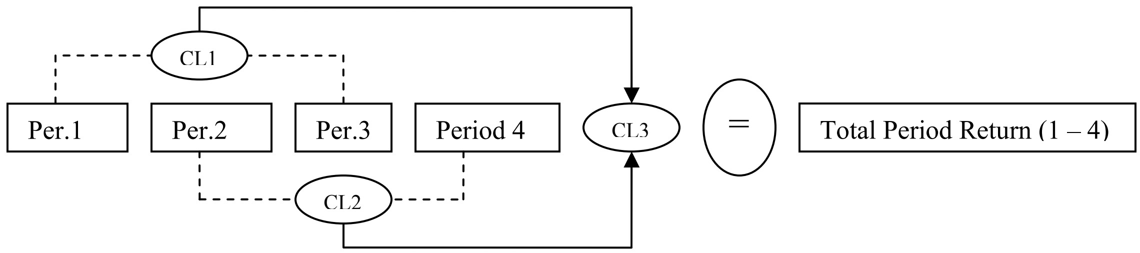
This article introduces the concept of consistent linking and algorithms for implementing this concept. Consistent linking is functionally similar to geometric linking. Geometric linking is used for calculating rate of return for the overall period based on sub-period returns. However, geometric linking generally produces a different result from rate of return calculated for the overall period considered as a single period. This makes impossible the usage of geometric linking for precise calculation of rate of return based on sub-period returns. Consistent linking, unlike geometric linking, always produces the same rate of return as if it was calculated for entire period as a single one. Rates of return for sub-periods have to be calculated only once and then can be used in any combination within longer periods. Article explores mathematical aspects of the problem and uses accordingly mathematical abstract model, thus neglecting some specifics of performance measurement business, which can be added when more specific problem to be considered.

Alexandre Chestopalov
Student at University of Toronto, Department of Mathematical Statistics. Also works as Mathematical Analyst at SegmentSoft Inc. for the last one year and a half. Alexandre has been working previously as System Administrator and Senior System Administrator from 1999 being responsible for the design, implementation and deployment of high performance Internet and production application servers. Winner of some contests in system design.

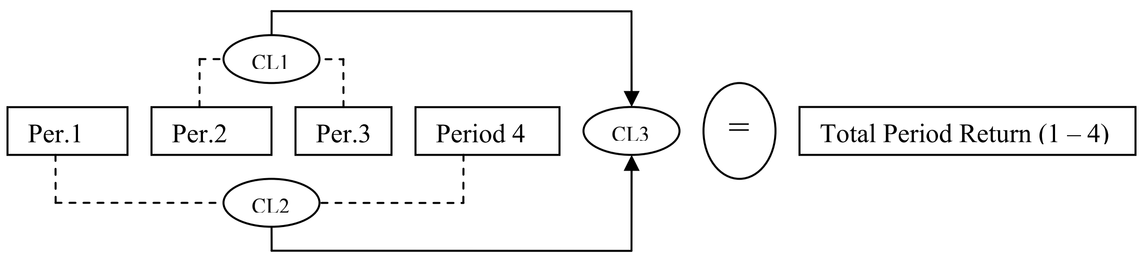
Konstantin Chestopalov
Student at University of Toronto, Department of Commerce and Finance. Previously has been working as System Administrator and Web Designer on part-time basis. Konstantin recently started to participate in analytical studies of methods improving performance of financial software applications, in particular investment performance measurement and trading systems.

Introduction

The picture below illustrates the essence of consistent linking. Letters CL inside the circle represent a consistent linking operation. Periods generally have different length.



Consistent linking will produce the same result for any other combination of periods. For example, result will be exactly the same if consistent linking first is used to link returns for periods 1 and 4, then for periods 2 and 3 and then linking results of two previous calculations.



Also, consistent linking can be used for linking asset slices. Sketch below demonstrates this feature of consistent linking. Dashed arrows show the direction of linking returns. Whatever slices are chosen, the final result of linking slices' returns will be equal to return calculated either for the whole portfolio, or using consistent linking of separate period returns for the total portfolio. It is possible to combine consistent linking of both slices and periods. Numerical example will follow.

Asset 1 Period 1	Asset 1 Period 2	Asset 1 Period 3	Total for Asset 1 for all Periods using CL
Asset 2 Period 1	Asset 2 Period 2	Asset 2 Period 3	Total for Asset 2 for all Periods using CL
Asset 3 Period 1	Asset 3 Period 2	Asset 3 Period 3	Total for Asset 3 for all Periods using CL
Total Portfol. For Period 1	Total Portfol. For Period 2	Total Portfol. For Period 3	Total Portfolio for all periods

The implications of this remarkable feature of consistent linking extend far beyond linking only sequential period returns. Consistent linking is also applicable to linking of non-sequential periods (for example January, March, and April monthly returns), as well as linking rates of return across securities, sectors, regions, etc. Furthermore, consistent linking works across periods with different lengths. Thus it is possible to calculate the precise rate of return across “slices”, including slices taken in non-sequential periods with different lengths. Whatever combination of such slices is chosen, as long as there is no overlapping, the total rate of return calculated by linking rates of return for all slices will be always the same. This is a very valuable feature for performance measurement, financial analysis of investment portfolios and also for developing trading strategies. This feature is also extremely appealing for developing performance measurement standards and resolving compliance issues related to inclusion or exclusion of certain assets within certain periods into accounting reports. In essence, consistent linking allows finding precise rate of return for arbitrary subset of assets for any combination of periods.

Another area benefiting from the concept of consistent linking is the computational one. Consistent linking simplifies the computation of rate of return for longer periods or calculating cumulative rate of return. It requires significantly less data and system resources thus providing much better system performance because of the small volume of data to be processed (thousand times less). Experimental system, based on consistent linking, demonstrated excellent performance and implementation simplicity.

Article structure

There are two related parts in the article. The first summarizes and revises briefly the necessary relationship of money weighted rate of return on investment portfolio (also called internal rate of return - IRR) and time weighted rate of return (TWRR) described in [I.Chestopalov, S. Beliaev, 2004]. TWRR is adapted by AIMR as the industry standard. (Term “money weighted” instead of “dollar weighted” has been introduced in [Feibel, 2003].) Consistent linking relies on underlying method for calculating rate of return. That is why before turning to consistent linking algorithms underlying methods have to be investigated.

It is shown in [I. Chestopalov, S. Beliaev, 2004] that present industry standard TWRR is a direct mathematical derivative of IRR. That uncovers the current standard TWRR from a different prospective, because it used to be considered a standalone method. This close relationship of two methods paves the path towards unification of standards and development of new performance measurement approaches. There is a big need in industry to calculate money weighted rate of return (MWRR) based on IRR equation. However, mathematical problems impede most of the possible developments in this direction. The article pays great attention to calculating MWRR using approximate methods, analyzes issues of controlling accuracy of those methods. It also introduced approximate consistent linking for MWRR, thus paving the road to much broader usage of MWRR. Generally speaking the set of formulas and approaches for MWRR introduced in the article forms fairly solid and complete mathematical base to apply MWRR to different business scenarios. Still there are some less essential mathematical issues to be resolved for completeness, but none of them seem as really principal ones.

The second part introduces concept of consistent linking and its implementation for TWRR and MWRR (IRR).

Relationship of TWRR and IRR

Below we consider the relationship of internal rate of return (IRR) and time weighted rate of return based on Modified Dietz formulae [I. Chestopalov, S. Beliaev, 2004], [Spaulding, 2003].
There is some ambiguity in using TWRR term, because its usage depends on the business context. Below TWRR is better to be considered as synonym of Modified Dietz method to avoid ambiguity.

Case of discrete compounding is considered in this article. Results for continuous compounding can be derived in a similar way.

Below the notion of "atomic period" is used. This is a period to which rate of return is related to. For discrete compounding the following equation is true

$$B + \sum_{j=1}^{N} C_j (1+R)^{-T_j} - E(1+R)^{-T_{N+1}} = 0 \qquad (1)$$

where B – beginning market value; E – ending market value; C_j – cash flow; T_j – time from beginning of period until cash flow occurred or length of the overall period (T_{N+1}) measured in units of chosen atomic period, j=1,…N; R – IRR to be found, equal to return during one unit of time.

The equivalent form of equation (1) can be derived by multiplying both parts of equation by $(1+R)^{T_{N+1}}$. Equation (1) can be rewritten as follows:

$$E = B(1+R)^{T_{N+1}} + \sum_{j=1}^{N} C_j (1+R)^{Tj} \qquad (2)$$

where T_j is now time period from when cash flow occurred until the end of the period measured in units of chosen atomic period. T_{N+1} is the length of the overall period.

The terms of the sum in equation (2) are substituted by Taylor expansion using linear terms. First solution has been found using Taylor expansion at point R=0. Thus found solution $R=R_0$ is used as the next point for Taylor expansion. For R=0 and atomic period $T_{N+1} = 1$ equation (2) transforms to the following:

$$E = B(1+R_0) + \sum [C_j + C_j T_j R_0] \qquad (3)$$

Choosing atomic period equal to the whole period doesn't reduce the generality of analysis.
Solution of equation (3) is as follows

$$R_0 = \frac{E - B - \sum C_j}{B + \sum C_j T_j} \qquad (4)$$

Expression (4) is TWRR - Modified Dietz formulae. So, TWRR is a Taylor approximation of IRR method at point R=0. It means that IRR is a primary method and TWRR (Modified Dietz formulae) is its direct mathematical derivative. In literature these two methods are considered as two separate entities. Dietz offered his famous formulae empirically, intuitively; it was not derived from IRR equation. That's why two approaches (IRR and TWRR) are considered essentially as two independent methods. Shouldn't be anymore.

As a note, there is high availability of computational capacity these days at virtually any level of performance management business, sufficient to do performance evaluation based on IRR or its approximations producing practically exact value of IRR. Some thoughts to this issue and appropriate formulas are given below.

Next step is to derive approximation method for calculation of internal rate of return. Equation (2) rewritten using Taylor expansion at point $R=R_0$ with linear terms looks as follows:

$$E = B(1+R) + \sum_{j=1}^{N} C_j [(1+R_0)^{T_j} + T_j (1+R_0)^{Tj-1} (R - R_0)] \qquad (5)$$

where R is the approximate IRR to be found.

Solving equation (5) results in the following expression for approximate value of IRR

$$R = \frac{E - B - \sum_{j=1}^{j=N} [C_j (1+R_0)^{T_j} - C_j T_j (1+R_0)^{T_j - 1} R_0]}{B + \sum_{j=1}^{j=N} C_j T_j (1+R_0)^{T_j - 1}} \qquad (6)$$

The time period associated with rate of return R in (6) is an atomic period. Since we have chosen the overall period $T_{N+1}=1$, R corresponds to the overall period.

Equation (6) helps with two issues. The first one is evaluation of discrepancy between TWRR and IRR. Subtracting R_0 from the right side of (6) results in

$$D = \frac{E - B(1 + R_0(E,B,C,T)) - \sum_{j=1}^{j=N} C_j (1 + R_0(E,B,C,T))^{T_j}}{R_0(E,B,C,T)(B + \sum_{j=1}^{j=N} C_j T_j (1 + R_0(E,B,C,T))^{T_j - 1})} \qquad (7)$$

Discrepancy D depends on the cash flow, rate of return, transaction time and beginning and ending market value. It is possible to find specialized cases when some trend can be discovered. However, essentially (7) represents non-systematic error. Thus numerous studies exploring differences between these two methods do not make much sense unless a very specific and restricted case study is considered.

The second issue formula (6) helps with is finding approximate value of IRR with any required accuracy (actually for practical purposes it can be considered as precise value). Substituting R_0 by calculated value R will result in finding more accurate value of rate of return because point of approximation is located closer to exact solution. That is

$$R_{n+1} = \frac{E - B - \sum_{j=1}^{j=N} [C_j (1+R_n)^{T_j} - C_j T_j (1+R_n)^{T_j - 1} R_n]}{B + \sum_{j=1}^{j=N} C_j T_j (1+R_n)^{T_j - 1}} \qquad (8)$$

Equation (8) is just more obvious form of approximation than other numerical methods offer.
Two-three iterations beginning from first approximation taken as TWRR return provided accuracy 10^{-7} for all simulated business scenarios, having up to few hundred transactions within the period. Appendix A includes table comparing accuracy of TWRR, linear approximation and quadratic approximation with IRR for the same input data. Technically equation (8) is very easy to implement, meaning to program. To make it more useful, some research regarding convergence conditions to be done.

Convergence of approximate method for finding IRR

The following trick can be done to explore this issue. We'd like to explore the first and second derivatives of right side equation (2) as function of R when R > -1.

$$E = B(1+R)^{T_{N+1}} + \sum_{j=1}^{N} C_j (1+R)^{Tj}$$

Let the atomic period be very large. In this case even three terms of Taylor expansion will result in a very accurate presentation of the function. The more accuracy we need, the greater atomic period to be chosen. This way we don't loose any generality. Using three first terms results in following:

$$E(R) = B[(1+R_0)^{T_{N+1}} + T_{N+1}(1+R_0)^{T_{N+1}-1} * (R - R_0) + 0.5 * T_{N+1} * (T_{N+1} - 1)(1+R_0)^{T_{N+1}-2} * (R - R_0)^2] +$$
$$\sum_{j=1}^{N} C_j [(1+R_0)^{T_j} + T_j (1+R_0)^{T_j - 1} * (R - R_0) + 0.5 * T_j * (T_j - 1)(1+R_0)^{T_j - 2} * (R - R_0)^2]$$

First derivative is equal to:

$$E^{'}(R) = B[T_{N+1}(1+R_0)^{T_{N+1}-1} + T_{N+1} * (T_{N+1} - 1)(1+R_0)^{T_{N+1}-2} * (R - R_0)] +$$
$$\sum_{j=1}^{N} C_j [T_j (1+R_0)^{T_j - 1} + T_j * (T_j - 1)(1+R_0)^{T_j - 2} * (R - R_0)]$$

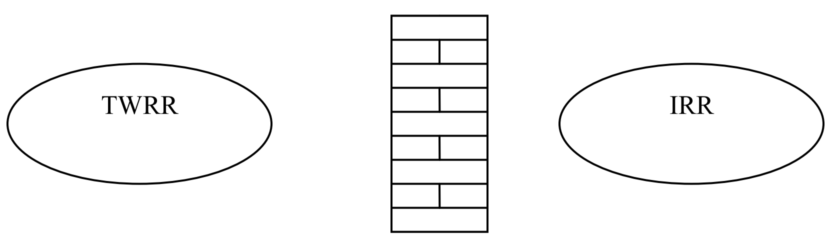
Second derivative is equal to

$$E^{''}(R) = B[T_{N+1} * (T_{N+1} - 1)(1 + R_0)^{T_{N+1}-2}] + \sum_{j=1}^{N} C_j [T_j * (T_j - 1)(1 + R_0)^{T_j - 2}]$$

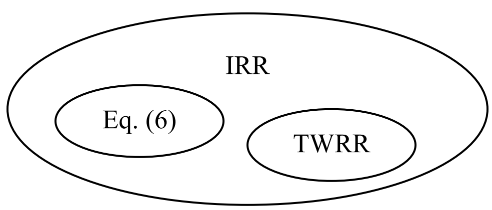
If the third term of Taylor expansion is neglected, then first derivative is constant regardless the value of R, so the function is a monotonic one. Taking into account the third term assures us at least there are no changes of first derivative sign from left or right of point R_0, which means at least uniform convergence of iterative procedure such as Newton's one to solution even if first derivative changes sign. However, if the accuracy of presentation is not sufficient, we can further increase atomic period until required accuracy is achieved with two terms only, in which case first derivative is constant and function is monotonic one. So, whatever initial approximation is chosen, it is always possible to choose atomic period large enough to achieve required accuracy with two terms of Taylor expansion, that means monotony of function within chosen range of R.

Constant value of second derivative assures that function either convex or concave for chosen range of R. Two conditions – monotony of function and its convexity or concavity, provide uniform convergence to true solution of iterative procedures used for solving IRR equation.

Discussion of results
Above results are important for understanding of current situation with TWRR as an industry standard. It can be pictured as follows; meaning TWRR and IRR are two different methods.



In reality TWRR as well as linear approximation (6) are subsets of IRR. The following picture is more appropriate.



Logically, precise method should be preferred as a standard. Modern computers allow doing this unlike forty years ago when Dietz introduced his formulae. However, besides accuracy the method for calculating rate of return should include other important features. A few of them are:

- Algorithm should be understood by users and have an acceptable level of complexity.
- Analytical form of algorithm is preferable for the purpose of analysis.
- Method can be applied to all variety of tasks related to performance measurement and optionally for developing trading strategies.
- Method should require reasonable computational resources and has to be computationally efficient.
- Can be relatively easily implemented.

From this standpoint it is not obvious that IRR is the best candidate. This is why approximate methods for calculating IRR cannot be ignored. For example, in the next part we'll see that TWRR has simple analytical algorithm for consistent linking, while IRR doesn't have analytical form of consistent linking at all.

Consistent Linking Algorithms

The main advantages of consistent linking algorithms are as follows.

- Rates of return for shorter periods together with a few other integral characteristics have to be calculated only once and then can be used in any combination within longer periods.
- Consistent linking allows linking together non-sequential periods having different length as well as different "slices" of investment portfolio across sequential and non-sequential periods. For any combination of non-overlapping periods and slices the total rate of return is always the same.
- Consistent linking algorithms facilitate implementation of performance measurement and analytical systems tremendously. Instead of one big set the data are organized into two subsets. The first one includes cash transactions, ending and beginning market value and transaction time to calculate integral characteristics of smaller periods (or slices). Then integral characteristics are used to compute rates of return for different combinations of periods and slices. Computational efficiency in this case is much better because set of integral characteristics is substantially (hundred and thousand times) smaller than the original data set.

Mathematical Formalization of Consistent Linking Algorithms

Mathematically the problem of finding consistent linking algorithms can be formulated as follows. Let $\mathbf{X} = \{\mathbf{x_1}, \mathbf{x_2}, \dots, \mathbf{x_n}\}$ be a primary dataset. Secondary dataset $\mathbf{G} = \{\mathbf{g_1}, \mathbf{g_2}, \dots, \mathbf{g_n}\}$ is some transformation of primary space $\mathbf{X}$ into space $\mathbf{G}$, $\mathbf{g_i} = \mathbf{F}(\mathbf{x_i})$. Dimension of vectors $\mathbf{g_i}$ and $\mathbf{x_i}$ can be different. Then function Φ provides consistent linking operation on a set of data $\mathbf{G} = \{\mathbf{g_1}, \mathbf{g_2}, \dots, \mathbf{g_n}\}$ if $\Phi(\mathbf{G}) = \mathbf{F}(\mathbf{X})$ for all possible subsets $\{\mathbf{x_1}, \mathbf{x_2}, \dots, \mathbf{x_k}\}$, $k > 0$ such that

$\mathbf{X} = \{\mathbf{x_1}, \mathbf{x_1}, \ldots, \mathbf{x_k}\}$. For example, addition and multiplication are trivial forms of consistent linking. Concept of consistent linking is not restricted by rates of return and can be applied to broad range of academic and practical problems.

Example:

Let $\mathbf{X} = \{x_1, x_2, \ldots, x_n\}$, $n<=N$, $g_k = \mathbf{F}(x_k) = x_k + x_k$, $k<=N$.

$\Phi(\mathbf{g_i}) = g_1 + g_2 + \ldots + g_k$; where $\mathbf{g_i}$ is a vector with components $\{ g_1, g_2, \ldots ,g_k \}$

Then $\Phi(\mathbf{G}) = \Phi(\mathbf{g_1}, \mathbf{g_2}, \ldots, \mathbf{g_k}) = \mathbf{F}(\mathbf{x_1}, \mathbf{x_2}, \ldots, \mathbf{x_k}) = \mathbf{F}(\mathbf{X})$ **and** $\Phi(\mathbf{G})$ provides consistent linking operation on a set of data $\mathbf{G}$ – simple addition of two elements from X space per each entry - g_m –th component of vector $\mathbf{g_j}$, $\mathbf{j}<=N$ instead of first calculating value of g_m and then do addition.

Consistent Linking for Internal Rate of Return

Let us consider two consecutive periods with length T_1 and T_2 having accordingly rates of return R_1 and R_2, and cash flows C_{1j}, C_{2j}. Ending market values can be expressed as follows using formulae (2):

$$E_1 = B_1(1+R_1)^{T_1} + \sum_{j=1}^{N} C_{1j}(1+R_1)^{T_{1j}} \qquad (9)$$

$$E_2 = B_2(1+R_2)^{T_2} + \sum_{j=1}^{N} C_{2j}(1+R_2)^{T_{2j}} \qquad (10)$$

Atomic periods can have different length in formulas (9) and (10), but for the validity of the following transformations they have to be measured in the same units of time (day, month, etc.). For sequential periods $B_2 = E_1$, formulae (10) can be rewritten in the following form

$$E_2 = [B_1(1+R_1)^{T_1} + \sum_{j=1}^{N} C_{1j}(1+R_1)^{T_{1j}}](1+R_2)^{T_2} + \sum_{j=1}^{N} C_{2j}(1+R_2)^{T_{2j}} \qquad (11)$$

Accordingly rate of return relates to chosen atomic period. Unlike in the above where the total period $T_{N+1}=1$, below a sub-period with length Tj is also measured in atomic periods. As a consequence the sub-period return must be calculated as $(R_{at}+1)^{T_j} - 1$, where R_{at} is return for atomic period.

For period N the general formulae is as follows

$$E_N = B_1\prod_{n-1}^{N}(1+R_1)^{Tn} + \sum_{n=1}^{N-1}\sum_{i=1}^{I_n} C_{ni}(1+R_n)^{T_{ni}} \prod_{k=n+1}^{N}(1+R_k)^{T_k} + \sum_{i=1}^{I_N} C_{Ni}(1+R_N)^{T_{Ni}} \qquad (12)$$

where I_n is the number of cash transactions within n-th period.

Formulae (12) can be proven using mathematical induction. Introducing additional function $P_N(R_n)$ simplifies (12).

$$P_N(R_n) = \prod_{i=n+1}^{N} (1+R_i)^{T_i} \quad \text{if n < N,}$$

$$P_N(R_n) = 1 \quad \text{if n = N}$$

$$E_N = B_1 \prod_{n=1}^{N} (1+R_n)^{T_n} + \sum_{n=1}^{N} \sum_{i=1}^{I_n} C_{ni} (1+R_n)^{T_{ni}} P_N(R_n) \tag{13}$$

(It is interesting to see from formulae (13) that geometric linking is a particular case of consistent linking - first term of equation is usual geometric linking. So, when there are no cash transactions, consistent linking becomes geometric linking. Thus we have a typical case when more general theory includes previous more particular method. In a theory of knowledge verification this is considered as one the signs of new theory validity.

For the entire period from the beginning of period one till the end of N-th period the following equation has to be true

$$E_N = B_1 (1+R_S)^{T_S} + \sum_{i=1}^{I_S} C_i (1+R_S)^{T_i} \tag{14}$$

where subscript S relates to the total period.

Equating right sides of (13) and (14) produces equation with unknown value R_S. It doesn't have analytical solution which is a function of sub-period IRR's rate of returns. So, IRR method doesn't have analytical consistent linking algorithm.

Important consequence of formulae (13) is as follows. Every entire sub-period can be considered as an atomic one because function $P_N(R_n)$ doesn't depend on the time scale. Each multiplier thus represents just rate of return for the entire sub-period. The same is true for the first term in (13). As it was mentioned, this term represents classic geometric linking of rates of returns. Thus the earlier introduced restriction regarding the same atomic period for all sub-periods can be discarded. This consideration is also applicable to all consistent linking algorithms considered below.

The value of the second term in (13) can be used to evaluate an error produced by geometric linking. The only case when geometric linking represents a true rate of return for the overall period is when the second term is equal to zero. It happens when a certain combination of time and value of cash transactions occurs.

<u>Consistent Linking for Time Weighted Rate of Return (Modofied Dietz)</u>

Using the same logic as in the previous subsection, but for underlying formulae (3), one can derive an equivalent of (13):

$$E_N = B_1 \prod_{n=1}^{N} (1+R_n) + \sum_{n=1}^{N} \sum_{i=1}^{I_n} C_{ni} (1+T_{ni} R_n) \overline{P_N}(R_n) \tag{15}$$

where

$$\overline{P_N}(R_n) = \prod_{i=n+1}^{N}(1+R_i) \quad \text{if n < N,} \tag{16}$$

$$\overline{P_N}(R_n) = 1 \qquad\qquad \text{if n = N}$$

Substituting (15) into (4) results in the following formulae for calculating rate of return for a total period.

$$R_{S0} = \frac{B_1[\prod_{n=1}^{N}(1+R_n)-1] + \sum_{n=1}^{N}\sum_{i=1}^{I_n}[C_{ni}(1+T_{ni}R_n)\overline{P_N}(R_n) - C_{ni}]}{B_1 + \sum_{i=1}^{i=\sum I_n} C_i T_i} \tag{17}$$

Equation (17) produces exactly the same value as Modified Dietz formulae applied to the whole period for the whole portfolio.
Equation (17) is based on (3). It means that each period is considered as an atomic one. This limitation does not affect first term of nominator in (17). However, there is no limitation that periods have to have the same length. Just unit of time the periods are measured in has to be same for all periods. For equal periods $T_i = (T_{ni} + (N-n))/N$. In this case equation (17) can be rewritten in the following form.

$$R_{S0} = \frac{B_1[\prod_{n=1}^{N}(1+R_n)-1] + \sum_{n=1}^{N}[S_{Tn}\overline{P_N}(R_n) - S_n]}{B_1 + \frac{1}{N}\sum_{n=1}^{N}[S_{Cn} + (N-n)S_n]} \tag{18}$$

where $S_{Tn} = \sum_{i=1}^{I_n} C_{ni}(1+T_{ni}R_n)$, $S_n = \sum_{i=1}^{I_n} C_{ni}$, $S_{Cn} = \sum_{i=1}^{I_n} C_{ni}T_{ni}$

If periods have different length, then

$$T_i = (L_n T_{ni} + L_S - \sum_{k=1}^{k=n} L_k)/L_S$$

where L_n, L_S are lengths of period *n* and the total period accordingly measured in the same units of time.

Second term of denominator in (18) in case of different period lengths is equal to

$$\frac{1}{L_S}\sum[L_n S_{Cn} + (L_S - L(n))S_n],$$

where $L(n) = \sum_{k=1}^{k=n} L_k$

The remarkable thing about formulae (18) is its practicality. Values S_{Tn} , S_{Cn} and S_n are associated with sub-periods only. They do not depend on any values related to other sub-periods. So, to calculate overall return one needs to know only these integral characteristics of each sub-period. This is the essence of consistent linking.

Sub-periods can be quite small because how's accurate the final rate of return is determined by computational precision, not by the method itself – method produces precise value. For example, if fund has 100 transactions a day and one wants to calculate rate of return for 20 years based on daily returns using (18), then values S_{Tn} and S_n should be calculated with relative accuracy 10^{-5} ($10^{-3}/\sqrt{365x20}$) and product $C_{ni}(1+R_n)$ accordingly with relative accuracy 10^{-6} in order to get an accuracy of final rate of return about 10^{-3} – the job for a fifteen years old PC.

Formulae (18) is also important for performance measurement systems. These systems process huge volume of data. Algorithm (18) can improve system performance dramatically because it is using substantially smaller set of input data than direct methods.

Numerical example illustrating consistent linking for TWRR method is shown below. It is based on simulated data for three monthly periods. First rates of return for each period were calculated using Modified Dietz formulae (4), then they were linked for a total period three months using consistent linking – formulae (18). Table 1 shows results of calculation. Data used for calculation presented in Appendix A_0.

Table 1. Comparison of direct calculation and consistent linking.

Period 1, Rate of return in %, Modified Dietz	Period 2, Rate of return in %, Modified Dietz	Period 3, Rate of return in %, Modified Dietz	Total period as one , direct calculation using Modified Dietz. Rate of return in %	Consistent linking for all three periods using formulae (18). Rate of return in %
154.726	12.434	45.6618	203.726	203.726

Table 1 shows that total rates of return calculated using direct calculation of time weighted rate of return and consistent linking based on formulae (18) are exactly the same. Geometric linking produces in this case 317.17 % that is far away from the 203.726 % found by direct calculation and consistent linking.

<u>Consistent linking for non-sequential periods</u>

Generalization of formulae (18) for non-sequential periods is fairly straightforward. Jump between end of previous period and beginning of the next one is counted as an additional cash transaction occurred at the beginning of a period. It can be proved as follows. Let $D_2 = B_2 - E_1$.

$$E_2 = (E_1 + D_2)(1 + R_2) + \sum_{j=1}^{N} C_{2j}(1 + T_{2j}R_2) = E_1(1 + R_2) + \sum_{j=0}^{N} C_{2j}(1 + T_{2j}R_2)$$

that is the same equation (3) used for derivation (18). The final equation for non-sequential and non-equal periods is as follows.

$$R_{S0} = \frac{B_1[\prod_{n=1}^{N}(1+R_n)-1] + \sum_{n=1}^{N}[(S_{Tn} + D_n(1+R_n))\overline{P_N}(R_n) - (S_n + D_n)]}{B_1 + \frac{1}{L_S}\sum_{n=1}^{N}[L_n(S_{Cn} + D_n) + (L_S - L(n))(S_n + D_n)]} \qquad (19)$$

where $D_n = B_n - E_{n-1}$ if n > 1, $D_n = 0$ if n = 1.

Formulae (19) is the main one in this article. Consistent linking algorithm for approximate IRR method is introduced in Appendix B. However, from practical point of view formulae (19) is of primary importance.

Example with "Slices"

This example uses data from table in Appendix C. The whole period is 18 days with each smaller period equal to 6 days. Portfolio includes three types of assets. Equation (19) has been used.

Table 2. Consistent linking of asset slices.

	Period 1	**Period 2**	**Period 3**	**Total**
Asset 1	1.77865612	0.22468793	0.0767590	2.4141176
Asset 2	-0.09836065	-0.2833675	0.0272108	-0.3270440
Asset 3	0.35294117	0.08759124	0.5703422	1.3469387
Total Portfolio	0.62730627	0.03238866	0.1425091	**0.8281631**
Accumulated return for the total portfolio	0.62730627 Period 1	0.625210084 Period 1 + Period 2	0.8281631 Period 1 + Period 2 + Period 3	

Dashed arrows show the directions of consistent linking. Rates of return can be linked both horizontally (linking period returns for the same asset type), and also vertically (for all assets within the same period). Total results for each asset are finally also linked. It is seen that either linking is done for the whole portfolio for three periods, or first for each asset for three periods and then for the whole portfolio across returns per asset, final result **0.8281631** is the same. It proves applicability of consistent linking for evermore sophisticated analysis of different combination of asset slices across investment portfolio. Obviously, the same technique can be applied to analysis of trading strategies. Last row demonstrates consistent linking on cumulative basis for the whole portfolio.

Conclusion

Calculating rate of return on investment portfolios presently relies on different methods. These methods generally produce different results for the same portfolio. Using all these methods would be justified if each of them could bring additional business value. However, all of them calculate the same characteristic – rate of return. Unification of these methods will help to decrease ambiguity of performance measurement process. Choosing the most adequate method for calculating rate of return is a multivariate problem. It is not only the matter of accuracy. Other criteria are also important.

Iterative method for finding IRR without solving numerically IRR equation was proposed. This iterative method allows finding practically precise value of IRR within 2-3 iterations. The procedure robustly converges to true solution. Method also has consistent linking feature that can be used with predefined accuracy or iteratively to find IRR with required accuracy. Overall the article introduced the whole set of formulas and approaches that can be used to calculate practically precise IRR for any imaginable business case without recursion to direct solving of IRR equation.

This article introduced and implemented concept of consistent linking for Modified Dietz formula and approximate IRR method. Consistent linking is very beneficial approach, allowing producing the precise value of rate of return by linking rates of return for both sequential and non-sequential periods with different length, as well as for portfolio "slices". It should be considered as one of the mandatory criteria when defining a method for calculating rate of return. It was shown that TWRR has consistent linking feature, while there is no analytical consistent linking algorithm for IRR. At the same time it was demonstrated that IRR is a primary method TWRR is directly derived from. IRR is the most objective way of calculating rate of return.

Below there is a brief comparison of three methods studied in the article rated on subjective authors' scale from 1 to 10. More criteria can be applied depending on particular requirements.

Feature\ Method	**IRR**	**Approximate IRR (6)**	**TWRR**
Accuracy	9	9	7
Average users can understand algorithm and accept this level of complexity	5	8	9
Analytical presentation of algorithm	1	8	10
Can be applied to all variety of tasks related to performance measurement and optionally for developing trading strategies	3	9	10
Requires reasonable computational resources and computationally efficient	3	7	9
Easiness of implementation	3	7	9
Simplicity of consistent linking algorithm	0	8	10

The way to go authors foresee as follows. TWRR is an industry standard. Appending it with consistent linking feature will produce a very powerful standard adding enormous opportunities for investment analysis, portfolio optimization and monitoring including real time monitoring, optimization and portfolio rebalancing. Being appended to existing standard, consistent linking can be used right away without any current standard changes. In terms of computational simplicity and efficiency this is also the simplest way to go.

At the same time the set of proposed approaches and formulas mathematically allows to switch to IRR completely for any foreseeable business case. Limitations are in business practice, not in mathematics or implementation. Approximate consistent linking for IRR also exists. It is more complicated than in case of Modified Dietz formulae, but accuracy is well controlled and except complexity (fairly reasonable from authors prospective) there are no other drawbacks in its potential usage.

ACKNOWLEDGEMENT

Authors would like to express their gratitude to Dr. I.Chestopalov for mentoring the whole project from the very beginning setting forth the problem and idea of consistent linking itself, until the very end helping with numerical verification of derived formulae.

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Appendix A_0

Table A_0-1. Simulated data for three consecutive periods

Period 1			Period 2			Period 3		
Cash Trans.	Transac-tion Date	Market Value	Cash Trans.	Transac-tion Date	Market Value	Cash Trans.	Transac-tion Date	Market Value
0	0	123	0	0	525	0	0	935
-15	1	120	15	1	520	-25	1	920
40	3	180	40	3	580	40	3	980
10	4	210	10	4	610	-10	4	1010
-26	6	206	26	6	606	-26	6	1006
3	7	220	3	7	620	-3	7	1020
7	8	220	7	8	620	7	8	1020
2	11	230	2	11	630	2	11	1030
3	15	220	3	15	620	3	15	1020
20	17	240	20	17	640	20	17	1040
10	18	260	10	18	660	-10	18	1060
15	19	270	15	19	670	-15	19	1070
3	20	290	3	20	690	3	20	1090
20	22	310	20	22	710	20	22	1110
16	24	350	16	24	750	16	24	1150
-33	26	365	33	26	765	33	26	1165
40	28	400	40	28	800	-40	28	1200
11	29	450	11	29	850	11	29	1250
25	29	490	25	29	890	-25	29	1290
-20	30	520	20	30	920	-20	30	1320

Appendix A

Accuracy of different methods for calculating rate of return

Table below is using results from [Chestopalov I., Beliaev S., 2004]. However, the results for column 5 were recalculated using as first approximation time weighted rate of return (R_0). That's why the actual results are better than in the cited article.

Table 1A: Differences in rates of return for different methods compared to IRR.

1 Average period return	2 Standard Deviatio n of period returns	3 Cash Flow / Beg. Market Value	4 IRR (true internal rate of return) formulae (2)	5 IRR minus IRR linear Approxi mation ROR, (2)-(6)	6 IRR minus Modified Dietz ROR, (2)-(4)	7 IRR minus Geometri c linking	8 IRR minus (2A) with R_0 from (4)
0.09	0.03	1.12	0.82228	0.00000	0.02860	0.05062	0.00000
0.04	0.08	0.96	0.25169	0.00000	0.00296	-0.01644	0.00000
0.12	0.11	0.09	1.20685	0.00000	0.00685	-0.00536	0.00000
0.05	0.05	0.09	0.43188	0.00000	0.00111	0.00096	0.00000
0.21	0.20	-0.15	2.27982	-0.00001	-0.05351	-0.18698	0.00000
0.07	0.04	-0.07	0.64333	0.00000	-0.00220	-0.00145	0.00000
0.10	0.11	0.31	0.82055	0.00000	0.00923	-0.02813	0.00000
-0.14	0.23	0.96	-0.72084	0.00001	0.04566	0.00338	0.00000
-0.02	0.07	3.06	-0.23563	0.00000	0.00621	-0.11418	0.00000
0.00	0.01	-0.80	-0.0753	0.00000	0.00086	-0.04197	0.00000

Table 1A shows that linear approximation method is a very adequate one for calculation rate of return. The maximum error is of the order of 10^{-3} % (column 5). Errors introduced by Modified Dietz method (TWRR) are in the range 0.1 – 5.3 % (column 6). Geometrical linking is a champion in producing the biggest errors (range 0.1-18 %, column 7). Column 8 shows that accuracy of method based on (1A) is outstanding when first approximation is used. Discrepancy was less than 10^{-5} %.

Appendix B.

Consistent Linking for Approximate IRR Method

Consistent linking for the underlying method (7) is done in a very similar way. The result for equal periods will be the following

$$R_S = \frac{B_1(\prod_{n=1}^{n=N}(1+R_n)-1)+\sum_{n=1}^{N}\{\sum_{i=1}^{I_n}C_{ni}[(1+R_{n0})^{T_{ni}}+(R_n-R_{n0})\sum_{i=1}^{I_n}T_{ni}(1+R_{n0})^{T_{ni}-1}]\}P_N(R_n)-}{B_1+\sum_{n=1}^{N}(1+R_{S0})^{-\frac{n}{N}}\sum_{i=1}^{i=I_n}(1+R_{S0}\frac{n}{N}-R_{S0}\frac{T_{ni}}{N})(1+R_{S0})^{\frac{T_{ni}}{N}}}$$

$$\frac{-\sum_{n=1}^{N}(1+R_{S0})^{-\frac{n}{N}}(\frac{1}{N}\sum_{i=1}^{i=I_n}C_{ni}T_{ni}(1+R_{S0})^{\frac{T_{ni}}{N}}+\frac{N-n}{N}\sum_{i=1}^{i=I_n}C_{ni}(1+R_{S0})^{\frac{T_{ni}}{N}})}{}$$

(1B)

where $P_N(R_n)$ is the same as in formulae (18)

There is no direct way to extract integral characteristics associated with sub-periods only. Approximate solution can be found expanding expression $(1+R_{S0})^{\frac{T_{ni}}{N}}$ within point zero for the power $\frac{T_{ni}}{N}$. Value 0 < T_{ni} < 1, N > 1. So, 0 < $\frac{T_{ni}}{N}$ < 1. Required accuracy will determine the number of terms to be taken into account.

After aforementioned transformations (20) can be rewritten as follows.

$$R_S = \frac{B_1(\prod_{n=1}^{n=N}(1+R_n)-1)+\sum_{n=1}^{N}\{S_{Rn}(0)+(R_n-R_{n0})S_{Rn}(1)\}P_N(R_n)-}{B_1+\sum_{n=1}^{N}(1+R_{S0})^{-\frac{n}{N}}\{\sum_{k=0}^{K}\frac{\ln^k(1+R_{S0})}{k!N^k}[\frac{1}{N}S_{Tn}(k+1)+\frac{N-n}{N}S_{Tn}(k)]\}}$$

$$\frac{-\sum_{n=1}^{N}(1+R_{S0})^{-\frac{n}{N}}\{\sum_{k=0}^{K}\frac{\ln^k(1+R_{S0})}{k!N^k}[(1+R_{S0}\frac{n}{N})S_{Tn}(k)-\frac{R_{S0}}{N}S_{Tn}(k)]\}}{}$$

(2B)

where

$$S_{Rn}(k)=\sum_{i=1}^{I_n}C_{ni}T^k{}_{ni}(1+R_{n0})^{T_{ni}-k}$$

$$S_{Tn}(k)=\sum_{i=1}^{I_n}C_{ni}T_{ni}^k$$

Value K depends on the required accuracy. For example, for K=4, N=10 accuracy of the third term in the denominator is about 10^{-8}.

Formulae (2B) is tied to formulas (18) and (19). Both formulas rely only upon values associated with sub-periods (R_{s0} eventually is calculated based on sub-periods values only, so there is no contradiction in the above statement). Once sub-period values are calculated, they always will be the same regardless the length of the total period.

Formulae (2B) proves that it is possible to link sub-period returns calculated using approximate IRR method into return for the overall period. Data associated with sub-periods are invariant to the length of the total period.

The other feature of equation (2B) relates to value of initial rate of return. Equation (8) doesn't require using TWRR as the previous iteration value. The only limitation is that R_{S0} has to be derived from composition of R_{n0}. We know how to do this for time weighted rate of return, that's the only reason why it is in the formulae (2B). However, the interesting thing is that the iteration logic applied to equation (8) works in this case too. So, R_{S0} can be used as a first approximation to find more precise value R_{S1} (next approximation). Then R_{S1} to be used instead of R_{S0} and so forth - R_{Sn} converges to value of internal rate of return for the whole period (limitation is imposed by accuracy of Taylor representation of $(1+R_{S0})^{\frac{T_{ni}}{N}}$ term).

We don't know appropriate values R_{n0} composing the next iteration value for the total period R_{Sn}. However, this problem is solved by assuming $R_n = R_{n0}$. This assumption doesn't influence the final result because R_n converges to R_{n0} as long as R_{Sn} converges to internal rate of return that formulae (8) does.

Appendix C

Data used for analysis of asset slices across portfolio.

Beginning Market Value		Cash Flow	Date	Ending Market Value
Total Portfolio				
Period 1	128	-20	1/1/2004	120
	120	40	1/2/2004	180
	180	10	1/3/2004	210
	210	-26	1/4/2004	206
	206	3	1/6/2004	220
Period 2	220	2	1/7/2004	230
	230	3	1/8/2004	220
	220	20	1/9/2004	240
	240	10	1/10/2004	260
	260	15	1/11/2004	270
	270	-5	1/12/2004	273
Period 3	273	5	1/14/2004	275
	275	3	1/16/2004	280
	280	-30	1/18/2004	290
Asset 1				
Period 1	60	-15	1/1/2004	60
	60	10	1/2/2004	80
	80	-5	1/3/2004	110
	110	-16	1/4/2004	100
	100	1	1/6/2004	110
Period 2	110	1	1/7/2004	115
	115	-1	1/8/2004	120
	120	10	1/9/2004	135
	135	2	1/10/2004	140
	140	10	1/11/2004	150
	150	-6	1/12/2004	153
Period 3	153	5	1/14/2004	155
	155	5	1/16/2004	160
	160	-20	1/18/2004	155
Asset 2				
Period 1	40	-10	1/1/2004	35
	35	10	1/2/2004	60
	60	10	1/3/2004	65
	65	10	1/4/2004	60
	60	5	1/6/2004	60
Period 2	60	6	1/7/2004	67
	67	8	1/8/2004	55
	55	5	1/9/2004	61
	61	5	1/10/2004	70
	70	7	1/11/2004	70
	70	2	1/12/2004	70
Period 3	70	1	1/14/2004	72
	72	7	1/16/2004	70
	70	-5	1/18/2004	75
Asset 3				
Period 1	28	5	1/1/2004	25

	25	20	1/2/2004	40
	40	5	1/3/2004	35
	35	-20	1/4/2004	46
	46	-3	1/6/2004	50
Period 2	50	-5	1/7/2004	48
	48	-4	1/8/2004	45
	45	5	1/9/2004	44
	44	3	1/10/2004	50
	50	-2	1/11/2004	50
	50	-1	1/12/2004	50
Period 3	50	-1	1/14/2004	48
	48	-9	1/16/2004	50
	50	-5	1/18/2004	60